To each real function $f$ defined on a Riemann surface $W$ and having a.e. partial derivatives which belong locally to $L^2$, may be associated a Borel measure $\mu_f$ on $W$ so that for each local map $(V,h)$ and for each Borel set $B \subseteq V$ the measure $\mu_f(B)$ coincides with

$$ \int_{h(B)} ||\text{grad } f \circ h^{-1}||^2 dx \, dy $$

This integral not depending on $h$ will be denoted by

$$ \int_{B} ||\text{grad } f||^2(x,y) dx \, dy. $$

It has a meaning even if $B$ is not included in a parametric set $V$ (see [1]). Such a measure $\mu_f$ is called a measure on the Riemann surface $W$ corresponding to the function $f$.

Let $W$ and $W'$ be two Riemann surfaces and let be $\phi: W \rightarrow W'$ a quasiconformal mapping. For the real function $f$ defined on $W$ and having a.e. partial derivatives which belong locally to $L^2$, we denote by $f' = f \circ \phi^{-1}$.

The analytic characterization of the quasiconformality, [2], assures us that $f'$ has also partial derivatives a.e. on $W'$ which belong locally to $L^2$.

Let $\mu_{f'}$ be the measure on $W'$ corresponding to the function $f'$. The measures $\mu_f$ and $\mu_{f'}$ are called associated measures by means of the quasiconformal mapping $\phi$.

We proposed to study the relationship between these measures and we found that they verify a well-known type of inequality.

**Theorem:** If $\mu_{f'}$ is associated to $\mu_f$ by means of the $K$-quasiconformal mapping $\phi$, then for each Borel set $B \subseteq W$ we have

$$ \frac{1}{K} \mu_f(B) \leq \mu_{f'}(\phi(B)) \leq K \mu_f(B) $$

**Proof.** We suppose that $\phi$ may be written by means of a local parameter in the form

$$ \phi(x,y) = x'(x,y) + iy'(x,y), $$

where $z = x + iy$ is a point of regularity of $\phi$.  


Let \( J(\phi(x,y)) \) be the Jacobian matrix of \( \phi \). Let us denote the matrix \( \overline{J}_\phi \) by

\[
\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{12} & a_{22}
\end{pmatrix}.
\]

We suppose that \( f' \) has partial derivatives at the point \( (x', y') \) and we put \( \partial f'/\partial x' = X, \partial f'/\partial y' = Y \).

We note first that \( \text{grad } f = J_\phi \text{grad } f' \), hence

\[
\|\text{grad } f\|_2^2 = (\text{grad } f)(\text{grad } f') = (\overline{J}_\phi \text{grad } f')(\overline{J}_\phi \text{grad } f')
\]

\[
= \text{grad } f' \overline{J}_\phi J_\phi \text{grad } f' = (X, Y)\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{12} & a_{22}
\end{pmatrix}^{(X)}
\]

\[
= a_{11}X^2 + 2a_{12}XY + a_{22}Y^2.
\]

The last expression represents a positively defined quadratic form, as follows by applying the Schwarz inequality to the

\[
\left( \frac{\partial x'}{\partial x}, \frac{\partial x'}{\partial y} \right) \text{ and } \left( \frac{\partial y'}{\partial x}, \frac{\partial y'}{\partial y} \right).
\]

Hence the preceding relation represents the equation of an ellipse.

Let \( a(x, y) \) and \( b(x, y) \) be its semi-axes. Then:

\[
b^2(x^2 + y^2) \leq a_{11}x^2 + 2a_{12}XY + a_{22}y^2 \leq a^2(x^2 + y^2)
\]

and hence

\[
b^2(\|\text{grad } f'\|_2^2 \circ \phi) \leq \|\text{grad } f\|_2^2 \leq a^2(\|\text{grad } f'\|_2^2 \circ \phi).
\]

By integrating these inequalities over \( B \) we obtain

\[
\int_B b^2(x, y)(\|\text{grad } f'\|_2^2 \circ \phi)(x, y)dx \, dy \leq \mu(B)
\]

\[
\leq \int_B a^2(x, y)(\|\text{grad } f'\|_2^2 \circ \phi)(x, y)dx \, dy.
\]

According to a classical result, the K-quasiconformality of \( \phi \) may be expressed by the following inequalities which take place in \( W \):

\[
\frac{1}{K} |J_\phi(x, y)| \leq b^2(x, y) \leq a^2(x, y) \leq K |J_\phi(x, y)|
\]

Bearing in mind these inequalities, we obtain for example in the right-hand part of the preceding inequalities:

\[
\mu(B) \leq \int_B a^2(x, y)(\|\text{grad } f'\|_2^2 \circ \phi)(x, y)dx \, dy.
\]