ON AN ABSTRACT VOLterra EQUATION

SERGIU AIZICOVICI
Institute of Mathematics
University of Iaşi
6600 Iaşi, Romania

1. INTRODUCTION

In this note we study the existence of solutions to a class of Volterra integrodifferential equations of the form

\[ u'(t) + \int_0^t a(t-s)g(u(s))ds \geq f(t), \quad 0 < t < T. \]

Here \( T \in (0, \infty) \) is arbitrary, \( u \) and \( f \) take values in a real infinite dimensional Hilbert space \( H \), \( a \) stands for a scalar convolution kernel, while \( g \) denotes a nonlinear monotone (possibly multivalued) operator acting in \( H \). (See \cite{3} and \cite{5} for background material on monotone operators).

To realize the difficulty of this problem, let us remark that in the case when \( a=1 \), \((1.1)\) formally reduces (by differentiation) to a nonlinear hyperbolic equation.

When approaching the existence of solutions to Eq.\((1.1)\), one has to choose between two opposite ways. The first way (used by Londen \cite{7},\cite{8}) rests upon hard conditions on the convolution kernel, excluding the case \( a=1 \) and therefore an application to hyperbolic equations. The second alternative (cf.\cite{1},\cite{2}) does allow a broad class of kernels (including \( a=1 \)), at the expense of strong restrictions on the admissible nonlinearities \( g \). We are going to further illustrate this second way.

2. MAIN RESULT

Consider a real reflexive, separable Banach space \( V \), such that \( V \subset H \), with dense and continuous inclusion. We have

\[ V \subset H \subset V', \]

where \( V' \) is the dual of \( V \). The pairing between \( v_1 \in V' \) and \( v_2 \in V \) will be denoted by \( (v_1, v_2) \); it coincides with their inner product in \( H \), whenever \( v_1 \in H \). We use the notations \( ||.||_V \) and \( ||.||_H \) to indicate the norms in \( H \) and \( V \), respectively. Assume that

\[ (2.1) \quad \text{The injection } V \subset H \text{ is compact.} \]

Let \( A \) be a cyclically maximal monotone operator in \( V \times V' \). Hence,
there exists a convex, lower semicontinuous (l.s.c.) and proper function \( \varphi : V \rightarrow (-\infty, +\infty] \), such that

\[
(2.2) \quad A = \partial \varphi, \quad (\partial = \text{subdifferential}).
\]

We suppose that

1. \( A \) is everywhere defined (\( D(A) = V \)), single-valued and maps bounded subsets of \( V \) into bounded subsets of \( V' \),

2. \( A \) is weakly continuous, i.e., for any sequence \( \{u_n\} \subset V \), such that \( u_n \rightharpoonup u \) weakly in \( V \), we have \( Au_n \rightharpoonup Au \) weakly-star in \( V' \),

3. \( \lim_{\|u\| \rightarrow \infty} \varphi(u) = +\infty \).

**Remark 2.1.** (i) Conditions (2.2)-(2.5) are clearly satisfied by each linear positive, symmetric and coercive operator \( A : V \rightarrow V' \).

(ii) Let \( \Omega \) be a bounded subset of \( \mathbb{R}^n (n \geq 3) \), with smooth boundary. If \( H = L^2(\Omega), V = H^1_0(\Omega) \),

then it is immediate that (2.1)-(2.5) hold, provided that the (nonlinear) operator \( A \) be given by

\[
Au = -\Delta u + M u, \quad u \in V,
\]

where \( M : L^p(\Omega) \rightarrow L^q(\Omega), 2 \leq p \leq 2n/(n-2), 1/p + 1/q = 1, \)

is of the form

\[
(Mu)(x) = \beta(u(x)), \quad x \in \Omega, \quad u \in L^p(\Omega),
\]

with \( \beta : \mathbb{R} \rightarrow \mathbb{R} \) satisfying

\[
\beta \in C(-\infty, +\infty), \beta \text{ monotone}, \beta(0) = 0,
\]

\[
|\beta(r)| \leq c(|r|^{p-1} + 1), \quad c > 0, \quad r \in \mathbb{R}.
\]

Consider next a convex, l.s.c., proper function \( \psi : H \rightarrow (-\infty, +\infty] \) and define the maximal monotone operator \( B \) in \( H \) by

\[
(2.6) \quad B = \partial \psi
\]

Denote by \( D(\psi) \) the effective domain of \( \psi \) and suppose that

\[
(2.7) \quad \forall \psi \cap \text{int.} D(\psi) \neq \emptyset, \quad (\text{int.} = \text{interior}).
\]

**Remark 2.2.** It is obvious that (2.7) is fulfilled in the case in which \( \psi \) is the indicator function of a closed convex subset \( K \subset H \), with

\[
\forall \cap \text{int.} K \neq \emptyset.
\]

Let \( a : [0,T] \rightarrow \mathbb{R} \) satisfy (cf. [6, Cond. (a)])