PART I. ANALYTICAL AND PROBABILISTIC BASIS

CHAPTER II. BASIC OPERATOR THEORY

We will deal with functions of a real (or complex) variable $x$ and of $D = \frac{d}{dx}$. Thinking of a function $f(x)$ as a multiplication operator, we "recover" the function as $f(x)1$. We will denote, then, for an operator $B$, the operator composition by $B^0 f(x)$; and the application of $B$ to $f(x)$ by $Bf(x)$, that is, $Bf(x) = B^0 f(x)1$.

The functional calculus we use will be based on the exponential. We have the following:

**Proposition 1**: Knowledge of $e^{AB}$ is equivalent to knowledge of $f(B)$ for the following families $(f)$:

1. Polynomials
2. Analytic functions (around $0 \in \mathbb{C}$)
3. Schwartz space functions and tempered distributions.

**Proof**: For (1) calculate $B^n = (\frac{d}{dx})^n |_0 e^{AB}$. (2) follows by power series expansion. From (1) or (2) we recover $e^{AB}$ by power series.

Recall Schwartz space $S = \{ f : f \in C^\infty$ and $\int |x|^m f(x) = 0$, for $|x| \to \infty \}$ all $n,m > 0 \}$. $S^* = \{ \text{tempered distributions} \}$. Then, for example,

$$\delta(B) = \frac{1}{2\pi i} e^{iyB} dy$$

and generally for $f \in S$ or $S^*$,

$$f(B) = \int e^{iyB} f(y) dy.$$

**Remark**: We use the normalization $f(y) = \frac{1}{2\pi i} e^{-iyx} f(x) dx$.

As $a$ runs from $-\infty$ to $= e^{AB}$ forms a group of operators. We are tacitly assuming that a suitable domain exists. A basic means of computing, or defining, $e^{AB} f$ is as the solution $u$ to

$$\frac{\partial u}{\partial a} = Bu , \quad u(0) = f.$$

The observation that enables us to use a quantum-mechanical viewpoint is simply this. Assume that $Bl = 0$. Then if the operator $U$ satisfies

$$\frac{\partial U}{\partial a} = [B,U] = BU - UB , \quad U(0) = f,$$

we see that $U = e^{AB} f e^{-AB}$ and $u = U1$. So we can always consider the evolution equations determining exponentials as operator equations. Using exponentials as a basis for our functional calculus we can determine inverse operators too.
We define
\[ B^{-1} = \int_0^1 \frac{\lambda}{\chi} \, d\chi \quad \text{and the generating function (resolvent)} \quad \frac{1}{B-z} = \int_0^\infty e^{zy} B^{-1}dy \]

and the generating function (resolvent) \( \frac{1}{B} \) by
\[ e^{z} = \int_0^\infty e^{zy} B^{-1}dy. \]

We will denote the Heaviside function by \( \chi \). Thus,
\[ \chi(x) = B^{-1} \delta(x) = \int_0^\infty \frac{e^{iyx}}{iy} \, dy = \int_0^\infty e^{-y} B^{-1}dy. \]

HEISENBERG GROUP FORMULATION

Given any two operators \( R, S \) such that \([R, S] = 1\), and \( R^l = 0 \), we can establish a calculus. For clarity we denote our operators by \( D \) and \( x \), noting that \( R \to D \), \( S \to x \) establishes an isomorphism of the given \( (R, S) \) system and the familiar one. Our first theorem is the

**Generalized Leibniz Lemma (GIL)**

\[ g(D)^n f(x) = \sum_{k=0}^n \frac{f^{(n)}(x)g^{(n)}(D)}{n!} \]

**Remark:** This allows us to express all products with derivative operators on the right.

**Proof:**

**Step 1.**
\[ D^n \circ x = xD^n + nD^{n-1} \quad \text{for } n > 0. \]

**n = 1:** Definition \([D, x] = 1\).

**n = m+1:** Multiply \( D^m \circ x = xD^m + mD^{m-1} \) on the left by \( D \).

Then \( D^{m+1} \circ x = xD^{m+1} + (m+1)D^m \) follows.

**Step 2.**
Multiplying \( D^n \circ x = xD^n + nD^{n-1} \) by \( \frac{t^n}{n!} \) and summing yields
\[ e^{tD} \circ x = xe^{tD} + te^{tD} = (x+t)e^{tD}. \]

Induction immediately yields
\[ e^{tD} \circ x^n = (x+t)^n e^{tD} \]

and hence
\[ e^{tD} \circ x = e^x e^{stD}. \]

**Step 3.** Therefore
\[ e^{tD} \circ x^n = \sum_{n=0}^\infty \frac{t^n}{n!} x^n e^{stD} = \sum_{n=0}^\infty \frac{t^n}{n!} x^n e^{stD} \quad e^{tD}. \]