ON SINGULAR SELF-ADJOINT STURM-LIOUVILLE OPERATORS

Gerhard K. Kalisch

The interest that singular self-adjoint Sturm-Liouville operators present from the standpoint of operator theory in Hilbert space $\mathbb{H}$ consists in their universality: all cyclic self-adjoint operators (bounded or not) are unitarily equivalent to restrictions of singular self-adjoint Sturm-Liouville operators and conversely. To put the facts in their proper setting and perspective, we present a number of known results as well as some new ones whose detailed exposition will be published elsewhere.

We assume an acquaintance with symmetric operator theory [3]. We also assume an acquaintance with elementary singular Sturm-Liouville theory on $(0, \infty)$ [2,5]. To establish our notation and definitions, let $D_0 = D_0(S_0)$ be the domain of definition of the symmetric operator $S_0$ and $D_0^* = D_0^*(S_0^*)$ be the domain of definition of its adjoint $S_0^*$; we write $\text{ind}(S_0) = (j,k)$ for the defect index of $S_0$ with $j$ and $k$ non-negative integers or infinity. Our operators in the sequel will have defect index $(1,1)$ and will have no self-adjoint parts; i.e., non-trivial self-adjoint restrictions to reducing subspaces. The self-adjoint extensions $S(a)$ of $S_0$ are defined on domains $D(a)$ with $D_0 \subset D(a) \subset D_0^*$; $a \in A$ is a suitable parametrization of these self-adjoint extensions.

Our Sturm-Liouville operators, which are defined on $L_2(0, \infty) = L_2$ with Lebesgue measure, are induced by the action of a differential expression given by $Xf = -f'' + qf$ for $f$ in a suitable dense linear subset of $L_2$ where the real-valued function $q$ is locally $L_1$. We define the symmetric operator $L_0$ induced by $X$ on the domain $D_0 = \{ f \in L_2; f & f' \text{ abs. cont.} \; Xf \in L_2; f(0) = f'(0) = 0 \}$. The operator $L_0^*$ is induced by $X$ also; its domain of definition is $D_0^* = \{ f \in L_2; f & f' \text{ abs. cont.}; Xf \in L_2 \}$. The defect index of $L_0$ is $(1,1)$ or $(2,2)$, Weyl's limit point and limit circle case, respectively. Our interest centers on the first case only; the self-adjoint extensions of $L_0$ have only discrete spectrum in the second case. See [2,5] for details.

Thus let $L_0$ have defect index $(1,1)$; its self-adjoint extensions $L = L(q, \alpha)$
are determined by the subset $D(q,\alpha)$ of $D_0^\dagger$ determined by the boundary condition at 0:

$$\sin \alpha f - \cos \alpha f' = 0 \quad \text{with} \quad \alpha \in (0,\pi].$$

The spectral theorem for $L(q,\alpha)$ then takes this form: Let $\varphi(s,z) = \varphi_\alpha$ and similarly $\psi(s,z) = \psi_\alpha$ be solutions of $Xf = zf$ ($z \in \mathbb{C}$) with boundary conditions at 0 given by $\varphi_\alpha = \sin \alpha, \varphi'_\alpha = -\cos \alpha; \psi_\alpha = \cos \alpha, \psi'_\alpha = \sin \alpha$. Then there exists a unique Borel measure $\rho = \rho(q,\alpha)$ on $\mathbb{R}$ given by a nondecreasing function also called $\rho$, and an isometry $\mathfrak{F} = \mathfrak{F}(q,\alpha)$ of $L_2$ onto $L_2(\rho)$ given by

$$(\mathfrak{F}f)(\lambda) = \int_0^\infty \psi(s,\lambda) f(s) \, ds, \quad (\mathfrak{F}^{-1}f)(s) = \int_{-\infty}^\infty \varphi(s,\lambda) F(\lambda) \, d\rho(\lambda).$$

If the self-adjoint operator $M$ on $L_2(\rho)$ is defined by $(MF)(\lambda) = \lambda F(\lambda)$ with domain of definition $D(M) = \{ F \in L_2(\rho); \; \lambda F \in L_2(\rho) \}$, then:

$$\mathfrak{F}D(q,\alpha) = D(M), \quad \mathfrak{F}^{-1}(D(M)) = D(q,\alpha),$$

$$\mathfrak{F}L(q,\alpha) = M\mathfrak{F} \quad \text{and} \quad \mathfrak{F}^{-1}M = L(q,\alpha)\mathfrak{F}^{-1}.$$ 

Furthermore, while $\varphi$ and $\psi$ are not in $L_2$, there exists a function $m = m(z) = m(z; q, \alpha)$ with $\varphi + m \psi \in L_2$ and $m(z) = \int_{-\infty}^{\infty} (1/(\lambda - z))d\rho(\lambda) - \tan \alpha$ (for the cases $\tan \alpha = \infty$, see [2,5]). The function $m$ is also related to $\varphi$ and $\psi$ by the formula $m(z) = -\lim_{\alpha \to \infty} (\varphi(a,z)/\psi(a,z))$. Finally the measure $\rho$, called the spectral function or spectral measure of $L(q,\alpha)$, is expressible in terms of $m$ by means of the formula

$$\rho(\lambda) = (1/\pi) \lim_{\gamma \to 0} \int_0^\infty \int_0^\gamma \text{Im} \, m(x + iy) \, dx.$$ 

As an example, consider the case $q = \alpha = 0$ and call the corresponding spectral measure $\rho_0$. This measure is zero on the negative reals and equals $(2/\pi)\sqrt{\lambda}$ on the positive reals; the corresponding function $\psi$ equals $\cos(\sqrt{\lambda} \, s)$ and the corresponding isometry $\mathfrak{F}_0$ is then essentially the cosine transform.

The fact that the Sturm-Liouville operators under consideration are unitarily equivalent to operators "multiplication by the independent variable" shows that only cyclic operators are representable in this manner. The question then naturally