LOWER BOUNDS FOR THE ACCURACY OF
LINEAR MULTISTEP METHODS

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1. Introduction

We consider linear multistep methods (2) for solving initial value problems
(1) \( y' = f(t,y) \), \( y(0) \) given

where one has large differences in the time constants. In the classical theory [1], [7] one requests stability (= zero stability) and accuracy of the method for small stepsizes \( h \). It has however long been observed that in the presence of large differences in the time constants one has to consider stability for large steps too, see e.g. Gear [5]. This leads to the definition of the stability region \( S \). For measuring the accuracy of the method one uses classically error order and error constant which describe the first term of the asymptotic expansion of the error as \( h \) tends to zero. Hence these are concepts for small values of \( h \). If one applies methods with "large" stability regions to the above mentioned stiff systems stability allows to use a stepsize \( h \) for which the asymptotic error expansion is no longer adequate. In this case the concepts of error order and error constants are no longer appropriate. We shall measure the accuracy for fixed positive \( h \) by the \( L_1 \)-norm of the Peano-kernel of the error functional. This allows to compare the accuracy of methods of different error order. The main result which we announce in this paper is, loosely speaking, the following. For methods of order higher than 2 the "accuracy" decreases as the "size" of the stability region increases. This can be considered as a refinement of the celebrated Dahlquist barrier [2] which says that the order of an A-stable method cannot exceed 2.

In section 2.1, 2.2 a short review of the classical theory is given. In section 2.3 the stability region and its relevance for stiff initial value problems is discussed. In section 2.4 we show that the concepts of error order and error constant are not appropriate in some situations. In section 3.1 we introduce Peano-kernels \( K_q \) for measuring the accuracy of a method for any \( h \). In section 3.2 we give our main result: lower bounds for the size of the Peano-kernel. These bounds depend on the radius \( R \).
of the largest disk $D_R = \{ u \in \mathbb{C} \mid |u+R| \leq R \}$ included in the stability region. In the last section we motivate why one requests $D_R \subset S$ and indicate how one could use the presented results to give estimates for the work which is needed to solve an initial value problem within a certain tolerance.

2. The methods and their application to stiff problems

2.1 Linear Multistep Methods

Let $h > 0$ be the stepsize and $t_n = nh$ for $n = 0,1,\ldots$. Then we compute recursively $y_{n+k}$ using

$$y_{n+k} = \sum_{i=0}^{k} a_i y_{n+i} + h \sum_{i=0}^{k} b_i f(t_{n+i}, y_{n+i}), \text{ for } n = 0,1,2,\ldots$$

Here we assume that the starting values already have been found and that $a_k \neq 0$. $y_n$ is an approximation to $y(t_n)$. If $b_k = 0$ then the method is called explicit, since (2) becomes a linear equation in $y_{n+k}$. However if $b_k \neq 0$ then (2) is in general a nonlinear equation and the method is called implicit. For non-stiff differential equations (2) is solved using an iteration scheme while for stiff equations one resorts to a Newton-like procedure.

Formulas of the form (2) have been derived in various ways, see e.g. Henrici [7] and Lambert [10]. For example one obtains the Adams-Moulton formulas by replacing the integrand $f(t,y(t))$ in the integral form of (1)

$$y_{n+k} - y_{n+k-1} = \int_{t_{n+k-1}}^{t_{n+k}} f(\tau, y(\tau)) \, d\tau$$

by the polynomial $P(\tau)$, which interpolates $f(\tau,y(\tau))$ at $\tau = t_n,t_{n+1},\ldots,t_{n+k}$. If $k = 1$ one obtains the trapezoidal rule

$$y_{n+1} - y_n = \frac{h}{2} \left( f(t_n,y_n) + f(t_{n+1},y_{n+1}) \right).$$

The so-called backward differentiation formulas (BDF) are derived by requesting that the interpolation polynomial $Q(t)$ with

$$Q(t_{n+i}) = y_{n+i} \text{ for } i = 0,1,\ldots,k$$

satisfies the differential equation (1) at $t_{n+k}$. 