Let $D$ be a principal ideal domain and $F$ be its field of fractions. Then it is well known that a linear representation of a finite group over $F$ is equivalent to a representation over $D$. This is proved by Burnside in an appendix to [1] for the case $D = \mathbb{Z}$ (the integers); and a complete proof appears, for example, in Theorem (75.3) of [3]. The object of this note is to prove a generalization for infinite groups.

**Theorem I.** Let $D$ be a principal ideal domain and $F$ its field of fractions. Let $\rho : G \to \text{GL}(n, F)$ be an irreducible representation of an arbitrary group $G$ over $F$. Suppose that there exists a finite normal separable extension $E$ of $F$ such that $\rho$ splits into absolutely irreducible representations over $E$. If for each $x \in G$, $\rho(x)$ is conjugate in $\text{GL}(n, F)$ to an element of $\text{GL}(n, D)$, then $\rho$ is equivalent to a representation $\sigma : G \to \text{GL}(n, D)$ over $D$.

**Remark I.** $\rho(x)$ is conjugate to an element of $\text{GL}(n, D)$ if and only if the eigenvalues of $\rho(x)$ are integral over $D$. The condition that each $\rho(x)$ have this property is clearly a necessary condition for the existence of a representation over $D$. In the case $G$ is finite, this condition is automatically satisfied.

2. In the finite case it is not required that $\rho$ should be irreducible, but some such condition is necessary in the general case. For example, if $G = (\mathbb{Q}, +)$ (the additive group of rationals), $D = \mathbb{Z}$, and $F = \mathbb{Q}$, then $\rho : G \to \text{GL}(2, \mathbb{Q})$ defined by

$$\rho(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is not equivalent to a representation over $\mathbb{Z}$.

3. It is not known whether the existence of the separable normal extension $E$ is a necessary hypothesis. But certainly this hypothesis holds in the cases:

(i) $\rho$ is absolutely irreducible; or

(ii) $F$ is perfect (see [3], §68).
In the case where $\rho$ is absolutely irreducible, an examination of the proof below shows that we do not need to assume each $\rho(x)$ is conjugate to an element of $\text{GL}(n, D)$, merely that the trace $\text{tr} \rho(x) \in D$.

**PROOF OF THE THEOREM.** Let $\theta = \theta_1$ be an irreducible constituent of $\rho^E = \rho \otimes_E E$, and let $\theta_1, \ldots, \theta_s$ represent the inequivalent conjugates of $\theta$ under the Galois group $\text{Gal}(E/F)$. Since $E$ is separable over $F$, $\rho^E = m_1 \theta_1 + \ldots + m_s \theta_s$ where each $\theta_i$ occurs as a constituent $m_i$ times (see [3], Theorem (70.15), or [4], Theorem 1.2 for more details). We can extend the definitions of $\rho$ and $\theta_i$ so that they are linear representations of the group algebra $F[G]$.

Define the $F$-linear functional $\tau : F[G] \to E$ by $\tau(a) = \sum_{i=1}^s \text{tr} \theta_i(a)$ where $\text{tr}$ denotes the trace. Note that the values of $\tau$ actually lie in $F$ since the $\theta_i$ are the conjugates of $\theta$ over $\text{Gal}(E/F)$. Moreover, if $x \in G$, then the eigenvalues of $\rho(x)$ are all integral over $D$ by hypothesis; therefore $\tau(x)$ is integral over $D$. Since a principal ideal domain is integrally closed in its field of fractions, this shows that $\tau(x) \in D$ for each $x \in G$. We also observe that if $a \in F[G]$ and $\tau(ax) = 0$ for all $x \in G$, then $\rho(a) = 0$. Indeed, since the $\theta_i$ are mutually inequivalent absolutely irreducible representations of $G$, a theorem of Frobenius and Schur implies that their coordinate functions are linearly independent (see [3] Theorem (27.8)). Thus $0 = \tau(ax) = \sum \text{tr} \theta_i(a) \theta_j(x)$ for all $x \in G$ implies $\theta_i(a) = 0$ for all $i$ and so $\rho(a) = 0$.

Now choose $x_1, \ldots, x_m$ in $G$ so that $\rho(x_1), \ldots, \rho(x_m)$ is an $F$-basis for the $F$-algebra spanned by $\rho(G)$. For each $x \in G$ we have $\lambda_j \in F$ such that

$$\rho(x) = \sum_{j=1}^m \lambda_j \rho(x_j).$$

Then, for each $k$, $\rho(x_k) = \sum_{j=1}^m \lambda_j \rho(x_j x_k)$ and so

$$\tau(x_k) = \sum_{j=1}^m \lambda_j \tau(x_j x_k) \quad \text{for } k = 1, \ldots, m.$$  

We claim that the $m \times m$ matrix $[\tau(x_j x_k)]$ is nonsingular. In fact, otherwise there would exist $a = \sum a_j x_j \in F[G]$ such that $\rho(a) \neq 0$ and $0 = \sum a_j \tau(x_j x_k) = \tau(x_k)$ for $k = 1, \ldots, m$. But this would imply $\tau(ax) = 0$ for all $x \in G$ because $Fp(G)$