Much progress in algebraic topology during the last twenty-five years has resulted from the discovery of homotopy equivalences (sometimes weak) of endofunctors of Top and Top*. Perhaps the first result of this kind was the suspension splitting of a product
\[(0.1) \quad \Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \# Y)\]
attributed to Peter Hilton [14]. This was followed by I.M. James's approximation \[X_\infty \simeq \rightbar X\] [9], the suspension splitting of \[\rightbar X\] [7] and more recently by the \[\rightbar^n\Sigma^n\] approximations due to Milgram [12] and May [13] and the splittings due to Snaith and others.

In an attempt to understand and hopefully to discover such equivalences the author introduced the concept of an X-functor (for a particular space X in Top*) motivated by the view that a natural transformation between X-functors ought to be a homotopy equivalence if it was so at X itself. Such considerations in the case \[X = \mathbb{P},\] the discrete two-point space, did indeed suffice to explain (0.1) and to prove others [6]. Although not recognized as such initially the X-functors were enriched left Kan extensions along the functor including the full subcategory of Top* with the single object X. At least this is the case if one replaces Top* by the convenient category C.G. Haus*. Now there are indications that one will have to consider Kan extensions along the inclusion of non-full subcategories. For example the configuration spaces that feature in May's approximations [13] are functorial only if one restricts to monomorphisms. Moreover they take values in C.G. Haus, not C.G. Haus*. Thus it seems that certain significant endofunctors can only be interpreted as relative Kan extensions and as Kan lift-extensions. In this paper the necessary concepts are formulated and a recognition principle given. There are no immediate applications, but the partial successes achieved in the simplest case [6], [5] lead one to
expect analogous results in due course.

1. Relative Kan extension and lift-extension.

Let $\Phi : \mathcal{V} \to \mathcal{V}'$ be a normal closed functor between symmetric monoidal closed categories. As pointed out [10;5], $\Phi$ induces a 2-functor $\Phi_* : \mathcal{V}\text{-Cat} \to \mathcal{V}'\text{-Cat}$. Let $S : \mathcal{A} \to \mathcal{B}$ be a $\mathcal{V}$-functor, let $J : \mathcal{X} \to \mathcal{A}$ and $\overline{S} : \mathcal{X} \to \mathcal{B}$ be $\mathcal{V}'$-functors and let $i : \overline{S} \to (\Phi_*S)J$ be a $\mathcal{V}'$-natural transformation. The pair $(S, i)$ is a (left) relative Kan extension of $S$ along $J$ if given any $\mathcal{V}$-functor $T : \mathcal{A} \to \mathcal{B}$ and any $\mathcal{V}'$-natural transformation $u : S \to (\Phi_*T)J$ there exists a unique $\mathcal{V}$-natural transformation $\overline{u} : \overline{S} \to T$ such that $(\Phi_*\overline{u})J.i = u$. In this case we denote $\overline{S}$ as usual by $\text{Lan}_J(\overline{S})$.

Now let $\overline{\mathcal{S}} : \mathcal{X} \to \mathcal{D}$ and $G : \mathcal{V}'\text{-Cat} \to \mathcal{D}$ be $\mathcal{V}'$-functors and let $j : \overline{\mathcal{S}} \to G(\Phi_*S)J$ be a $\mathcal{V}'$-natural transformation. The pair $(\mathcal{S}, j)$ is a (left) relative Kan lift-extension of $\overline{\mathcal{S}}$ along $J$ and up $G$ if given any $\mathcal{V}$-functor $T : \mathcal{A} \to \mathcal{B}$ and any $\mathcal{V}'$-natural transformation $u : \overline{S} \to G(\Phi_*T)J$ there exists a unique $\mathcal{V}$-natural transformation $u : S \to T$ such that $G(\Phi_*u)J.j = u$.

In the case that $G$ has a $\mathcal{V}'$-left adjoint $F : \mathcal{D} \to \mathcal{V}B\overline{\mathcal{S}}$ along $J$ the existence of a lift-extension is assured if a relative Kan extension of $F\overline{S}$ along $J$ exists.

**Proposition 1.** If $(\varepsilon, \eta) : F \to \mathcal{V}'G$ and if $(\text{Lan}_J(F\overline{S}), i)$ exists then $(\text{Lan}_J(F\overline{S}), j)$ is a (left) relative Kan lift-extension of $\overline{S}$ along $J$ and up $G$, where $j : \overline{S} \to G(\Phi_*S)J$ the unique arrow such that $i = \varepsilon(\Phi_*S)J.j$.

The proof of proposition 1 requires only elementary manipulation of the definitions.

2. Free objects.

A relative Kan extension can be interpreted as a free object in the metacategory $\mathcal{B}^A$ of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations. Let $R : \mathcal{V} \to \mathcal{W}$ be a functor between metacategories and let $j : W \to RU$ be an arrow in $\mathcal{W}$. We recall that $(U, j)$ is free over $W$ with respect to $R$ (in the