RELATIONS BETWEEN AUTOMORPHIC FORMS
PRODUCED BY THETA-FUNCTIONS

by

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1. GENERAL PHILOSOPHY

I want to begin with a few philosophical remarks on how theta-functions attached to indefinite quadratic forms can be expected to produce relations between modular forms and automorphic forms of one and several variables.

First it should be pointed out that this 'philosophy' is, in a certain sense, not new. It is certainly contained in Siegel's papers [7] on indefinite quadratic forms and function theory (1951/52). More recently, Shintani [10] and Niwa [5] have revived these ideas and have made a beautiful application to the theory of automorphic forms of \( \frac{1}{2} \)-integral weight.

Now suppose

\[ \mathcal{Q}^n = \text{an } n\text{-dimensional vector space over } \mathbb{Q} \]

and

\[ Q: \mathcal{Q}^n \to \mathbb{Q} \quad x \mapsto Q(x) = ^tXQX \]

is an indefinite quadratic form of signature \( (p,q) \). Take

\[ L \subset \mathcal{Q}^n \text{ a } \mathbb{Z}\text{-lattice such that } Q(L) \subset 2\mathbb{Z}. \]

If we want to construct a theta-function attached to \( Q \) and \( L \), we will also need a majorant \( R \) of \( Q \); that is: \( R \in M_n(\mathbb{R}) \), \( n = p + q \) such that (1) \( ^tR = R \), \( R > 0 \), and (2) \( RQ^{-1}R = Q \). For example, if \( Q \) is in diagonal form for some basis of \( \mathbb{R}^n \).
If we let

\[ \text{SO}(Q) = \{ g \in \text{SL}_n(\mathbb{R}) \text{ such that } ^t g Q g = Q[g] = Q \}, \]

and if \( R \) is one majorant, then every majorant has the form \( R[g] = ^t g R g \) for some \( g \in \text{SO}(Q) \). Let \( X \) be the space of all majorants of \( Q \). Then, by the preceding comment, \( X \) is the symmetric space attached to \( \text{SO}(Q) \), since \( \text{SO}(Q) \cap \text{SO}(R) \) is a maximal compact subgroup of \( \text{SO}(Q) \).

We can now make a theta-function:

\[
\theta(z,R) = \frac{q^{1/2}}{\sqrt{L}} \sum_{\ell \in L} e^{i \pi (uQ + ivR)[\ell]}
\]

which we view as a function of two variables, \((z,R) \in \mathcal{O} \times X\) where \( z = u + iv \in \mathcal{O} = \) upper half-plane.

For our purposes, the important fact about \( \theta \) is that it has a transformation law, like that of an automorphic form, in each variable. More precisely:

\[
(1) \quad \theta(\gamma z, R) = (cz+d)^{p-q} \theta(z, R)
\]

\( \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \) for suitable \( N \), e.g., \( N = 2 \text{det } Q \) will do.

If we let \( \Gamma_L = \{ U \in \text{SO}(Q) \text{ such that } UL = L \} = \) unit group of \( L \), then

\[
(2) \quad \theta(z, R[U]) = \theta(z, R).
\]

Remarks:

a) (2) is obvious from the definition of \( \theta \), while (1) follows from the usual Poisson-summation argument.

b) Notice that \( n = p+q = p-q \mod 2 \) so that, if \( n \) is even, \( \theta \)