DIFFUSION PROCESSES AND MARTINGALES 1*)

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Introduction

Let \( a: [0,\infty) \times \mathbb{R}^d \to S_d \) and

\( b: [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d \)

be bounded measurable functions, and form the elliptic operator

\[
L_t = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x) \frac{\partial}{\partial x_i}.
\]

Given \( s \geq 0 \) and \( x \in \mathbb{R}^d \), our aim is to prove the existence and uniqueness of a measure \( \pi_s,x \) which bears the same relation to \( L_t \) as the \( d \) dimensional Wiener \( \mathcal{W}_{s,x} \) measure, conditioned to start from \( x \) at time \( s \), bears to \( \frac{1}{2} \Delta \) ( \( \Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} \) ). We have succeeded in this program for the case when \( a \) is continuous and positive-definite valued.

Before proceeding, we must make precise what is meant by the relation of Wiener measure to \( \frac{1}{2} \Delta \). We will use the following notation:

\[
\Omega = C([0,\infty),\mathbb{R}^d),
\]

\( x_t(\omega) = x(t,\omega) \) is the position of \( \omega \in \Omega \) at time \( t \geq 0 \),

\( \mathbb{M}_t = \mathbb{M}[x_u: s \leq u \leq t], 0 \leq s \leq t, \) and \( \mathbb{M}_s = \mathbb{M}[x_u: u \geq s], s \geq 0. \)

*) This is a summary of the paper "Diffusion Processes with Continuous Coefficients I" by the authors. This paper appeared in Communications in Pure and Applied Mathematics. vol. XXII, 1969.

1) \( S_d \) is the class of \( d \times d \), non-negative definite, symmetric matrices.
In this notation, the Wiener measure $W_{s,x}$ is characterized as the probability measure on $<\Omega, \mathcal{F}, \mathbb{P}>$ satisfying

$$W_{s,x}(x(s) = x) = 1,$$

$$W_{s,x}(x(t_2) \in \Gamma \mid \mathcal{F}_{t_1}) = \int_{\Gamma} g(t_2 - t_1, x(t_1) - y) dy,$$

where

$$g(t, y) = \frac{(2\pi t)^{-d/2} e^{-|y|^2/2t}}{2}.$$

The problem with the formulation given in (I) is that it relates $W_{s,x}$ to $\frac{1}{2} \Delta$ through the fundamental solution $g(t, y)$ of the heat equation. Hence the analogous characterization of $P_{s,x}$ would have to involve the fundamental solution of the parabolic equation associated with $L_t$. For this reason, we will now give a second characterization of $W_{s,x}$. Let $f \in C_b^{1,2}([0, \infty) \times \mathbb{R}^d)$ and $A \in \mathbb{S}^d$, and consider

$$\Phi(u) = E^{W_{s,x}}[\chi_A f(t+u, x(t+u))].$$

Clearly,

$$\Phi(u) = E^{W_{s,x}}[\chi_A \int g(u, x(t) - y) f(t+u, y) dy],$$

and therefore

$$\frac{d}{du} \Phi(u) = E^{W_{s,x}}[\chi_A \int g(u, x(t) - y)(f_{t+u} + \frac{1}{2} \Delta f)(t+u, y) dy]$$

$$= E^{W_{s,x}}[\chi_A (f_{t+u} + \frac{1}{2} \Delta f)(t+u, x(t+u))].$$

From this we see that

$$E^{W_{s,x}}[f(t_2, x(t_2)) - f(t_1, x(t_1)) \mid \mathcal{F}_{t_1}]$$

$$= E^{W_{s,x}}[\int_{t_1}^{t_2} (f_{u} + \frac{1}{2} \Delta f)(u, x(u)) \mid \mathcal{F}_{t_1}],$$

2) $C_b^{1,2}([0, \infty) \times \mathbb{R}^d)$ is the space of bounded $f: [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ which have one bounded continuous time derivative and two bounded continuous space derivatives.

3) $E^P[\eta]$ denotes $\int_{\eta} dP$. 