few instances in which it will be convenient to use open covers. For example, if \( f, g : X \to Y \) are maps and \( \alpha \) is an open cover of \( Y \), then we say that \( f \) is \( \alpha \)-close to \( g \) provided that for every \( x \in X \) there exists an element of \( \alpha \) containing both \( f(x) \) and \( g(x) \). We say that \( f \) is \( \alpha \)-homotopic to \( g \), and write \( f \simeq_\alpha g \), provided that there exists a homotopy \( F : X \times I \to Y \) from \( f \) to \( g \) such that for every \( x \in X \) there exists an element of \( \alpha \) containing \( F([x] \times I) \). In the case that we are given a controlling map \( p : Y \to B \), then we write \( f \simeq_{p^{-1}(\varepsilon)} g \) to indicate that \( f \) is \( p^{-1}(\varepsilon) \)-homotopic to \( g \).

3. CONSTRUCTION OF \( \text{Wh}(Y)_{\varepsilon} \)

Throughout this section \( p : Y \to B \) will be a map of a compact polyhedron to a metric space. Our goal is to define the controlled Whitehead group \( \text{Wh}(Y)_{\varepsilon} \) and establish some of its elementary properties. We begin by defining \( \text{DR}(Y)_{\varepsilon} \) to be the collection of all PL maps \( f : X \to Y \) which are \( p^{-1}(\varepsilon) \)-sdr's, where \((X,Y)\) is a compact polyhedral pair. There is a natural addition on \( \text{DR}(Y)_{\varepsilon} \) which is defined as follows: for elements \( f_1 : X_1 \to Y \), \( f_2 : X_2 \to Y \) of \( \text{DR}(Y)_{\varepsilon} \) we define \( f_1 + f_2 : X_1 \cup X_2 \to Y \) by \( f_1 + f_2 | X_1 = f_1 \) and \( f_1 + f_2 | X_2 = f_2 \), where \( X_1 \cup X_2 \) is formed by sewing \( X_1 \) and \( X_2 \) together along \( Y \). Note that \( \text{id}_Y \in \text{DR}(Y)_{\varepsilon} \) satisfies \( f + \text{id}_Y = \text{id}_Y + f = f \), for all \( f \in \text{DR}(Y)_{\varepsilon} \).

For elements \( f_1 : X_1 \to Y \), \( f_2 : X_2 \to Y \) of \( \text{DR}(Y)_{\varepsilon} \) we define \( f_1 \simeq f_2 \) provided that there exists a compact polyhedron \( Z \) containing \( Y \) as a subpolyhedron and CE-PL maps \( r_1 : Z \to X_1 \) such that \( r_1 | Y = \text{id} \) and \( f_1 r_1 \simeq_{p^{-1}(\varepsilon)} f_2 r_2 \) rel \( Y \).

Clearly \( \simeq \) is a reflexive and symmetric relation, so it generates an equivalence relation \( \sim \) on \( \text{DR}(Y)_{\varepsilon} \). This means that \( f \sim f' \) provided that there exist elements \( f_1, \ldots, f_n \) such that \( f \simeq f_1 \simeq \cdots \simeq f_n \simeq f' \). Define \( \text{Wh}'(Y)_{\varepsilon} \) to be the set of all equivalence classes of this relation, i.e., \( \text{Wh}'(Y)_{\varepsilon} = \text{DR}(Y)_{\varepsilon}/\sim \).

For any \( f \in \text{DR}(Y)_{\varepsilon} \) we use \( [f]_{\varepsilon} \) to denote its equivalence class in \( \text{Wh}'(Y)_{\varepsilon} \).

Define an addition on \( \text{Wh}'(Y)_{\varepsilon} \) by \( [f_1]_{\varepsilon} + [f_2]_{\varepsilon} = [f_1 + f_2]_{\varepsilon} \), which is easily seen to be well-defined, associative, and commutative. Also \( 0 = [\text{id}_Y]_{\varepsilon} \) is an additive identity, so we conclude that \( \text{Wh}'(Y)_{\varepsilon} \) is a commutative monoid. Finally the controlled Whitehead group \( \text{Wh}(Y)_{\varepsilon} \) is defined to be the subgroup of \( \text{Wh}'(Y)_{\varepsilon} \) that
consists of all invertible elements of $Wh'(Y)_\mathfrak{c}$, i.e., elements $[f] \in Wh'(Y)_\mathfrak{c}$ for which there is an element $[f']$ that satisfies $[f]+[f'] = 0$.

Note that in the definition of the relation $\mathcal{E}$ given above we could have merely assumed that $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ are retractions. The maps appearing in the following result are of this type.

**Lemma 3.1.** If $f_1 \in \mathcal{E}$ and $f_2 \in \mathcal{E}$, then $f_1 \in \mathcal{E} \circ \mathcal{E}$.

**Proof.** Since $f_1 \in \mathcal{E}$, there is a compact polyhedron $Z_1$ and CE-PL maps $r_1 : Z_1 + X_1$, $r_2 : Z_1 + X_2$ such that $f_1 r_1 p^{-1}(\epsilon) = f_2 r_2$ rel $Y$. Similarly there is a compact polyhedron $Z_2$ and CE-PL maps $r'_2 : Z_2 + X_2$, $r_3 : Z_2 + X_3$ such that $f_2 r'_2 p^{-1}(\delta) = f_3 r_3$ rel $Y$. Now form the following diagram:

The space $Z$ at the top is a subpolyhedron of $Z_1 \times Z_2$ which results from a pull-back construction,

$$Z = \{(z_1, z_2) \mid r_2(z_1) = r'_2(z_2)\}.$$

$Y$ is identified with $\{(y, y) \mid y \in Y\} \subset Z$ and $u, v$ are projection maps which are CE-PL and which are the identity on $Y$. Finally we have

$$f_1 r_1 u = f_2 r_2 u = f_2 r'_2 v = f_3 r_3 v \text{ rel } Y.$$  

The above result seems to imply that the relation $\sim$ on $DR(Y)_\mathfrak{c}$ is unstable in the sense that all $\epsilon$-control is lost. This is definitely not the case, and the stability of the relation $\sim$ is the goal of the remainder of this section.

In the following result $X, W$ are compact polyhedra, $\alpha$ is an open cover of $W$, and $f_0, f_1 : X \to W$ are PL maps thus giving rise to polyhedral mapping