We define the Hamiltonian completion number of a graph $G$, denoted $hc(G)$, to be the minimum number of lines that need to be added to $G$ in order to make it Hamiltonian. The Hamiltonian completion problem asks for $hc(G)$ and a specific Hamiltonian cycle containing $hc(G)$ new lines. We derive an efficient algorithm for finding $hc(T)$ for any tree $T$, and show that if $S$ is the set of spanning trees of an arbitrary connected graph $G$, then

$$hc(G) = \min_{T_i \in S} hc(T_i).$$

A number of other general results are presented including an efficient heuristic procedure which can be used on arbitrary graphs.
ON THE HAMILTONIAN COMPLETION PROBLEM

1. INTRODUCTION

The Hamiltonian completion number of a graph $G$, denoted $hc(G)$, is the minimum number of lines which need to be added to $G$ to make it Hamiltonian. By definition, $0 \leq hc(G) \leq p$, where $p$ is the number of points in $G$.

The desirability of completing a graph in the above sense arises naturally in problems involving routing and the periodic traversal and updating of data structures.

The Hamiltonian completion problem is a special case of the traveling salesman problem in which each line in $G$ is assigned a weight of 0 and each line in $K_p$ not in $G$ is given a weight of 1. This relationship can be used to show that the problem is a member of Karp's polynomial complete class of difficult computational problems [1]. This approach to the completion problem is being pursued elsewhere.

An equivalent formulation of the completion problem for non-Hamiltonian graphs involves the notion of island decomposition. An island decomposition of a graph $G$ is an acyclic spanning subgraph of $G$ in which every point has degree $\leq 2$. An island decomposition is essentially a collection of point disjoint paths (islands) that cover the points of $G$. An isolated point can be a member of such a set. Let $i(G)$ denote the minimum number of islands and $l(G)$ the maximum number of lines in any island decomposition of $G$. Then it is not difficult to show the next statement.

**Lemma 1.** For any graph $G$ having $p$ points

1) $i(G) + l(G) = p$, and

2) either $G$ is Hamiltonian or $i(G) = hc(G)$.

**Lemma 1** shows that we can now Hamilton thread the $i(G)$ islands into a Hamiltonian cycle with the addition of $i(G)$ new lines. Note that this approach cannot distinguish between a Hamiltonian cycle and a Hamiltonian path in $G$. In both cases we get $i(G) = 1$, although $hc(G) = 0$ if $G$ has a Hamiltonian cycle. In what follows we assume $G$ is connected (the generalization is immediate).