A CONVERSE OF THE KUIPER-KUO THEOREM

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The aim of this note is to prove a converse of the following theorem due to N.H. Kuiper [1] and T.C. Kuo [2]:

An \( r \)-jet, represented by a polynomial \( w : \mathbb{R}^n \to \mathbb{R} \) (of degree \( \leq r \)), is \( C^r \)-sufficient in \( C^r \) if \( |\text{grad } w(x)| \geq c|x|^{r-1} \) holds in a neighbourhood of 0 with some \( c > 0 \).

The \( r \)-jet of a \( C^r \) function in a neighbourhood of 0 is identified with its \( r \)-th Taylor polynomial at 0 if \( r < \infty \) or with its Taylor series at 0 if \( r = \infty \); then the function is called a realization of the jet. An \( \omega \)-jet is an \( \infty \)-jet which is a convergent power series.

We say that an \( r \)-jet is \( C^p \)-sufficient (resp. \( \omega \)-sufficient) in \( C^k \), \( (r, p, k = 0, \ldots, \infty, \omega; r, p \leq k) \) if for any two of its \( C^k \) realizations \( \phi \), \( \psi \) there is a \( C^p \)-automorphism germ \( g \) of \( (\mathbb{R}^n, 0) \) such that \( \phi \circ g = \psi \) in a neighbourhood of 0 (resp. the germs of \( \phi^{-1}(0) \) and \( \psi^{-1}(0) \) at 0 are homeomorphic). Let \( v = \sum_{p} a_p x^p \) be an \( \omega \)-jet; put \( v_s = \sum_{|p| \leq s} a_p x^p \); we say that \( v \) is finitely \( C^p \)-(resp. \( \omega \)-)sufficient in \( C^k \), \( (k \geq \infty) \), iff \( v_s \) is \( C^p \)-(resp. \( \omega \)-)sufficient in \( C^k \) for some \( s < \infty \).

Then the result is the following

Theorem. Let \( v \) be an \( r \)-jet, \( \psi \) a \( C^r \) realization.
A. If \( r < \infty \) then the following conditions are equivalent:

(1) \( v \) is \( \omega \)-sufficient in \( C^r \),
(2) \( v \) is \( C^0 \)-sufficient in \( C^r \),
(3) \( |\text{grad } w(x)| \geq c|x|^{r-1} \) in a neighbourhood of 0, with some \( c > 0 \).
B. \textbf{If } r = \infty \textbf{ then the following conditions are equivalent:}

(1) \( v \) is \( V \)-sufficient in \( C^\infty \),
(2) \( v \) is \( C^0 \)-sufficient in \( C^\infty \),
(3) \( |\text{grad } w(x)| > c|x|^N \) in a neighbourhood of 0 with some \( c, N > 0 \)
(4) \( v \) is finitely \( V \)-sufficient in \( C^\infty \),
(5) \( v \) is finitely \( C^0 \)-sufficient in \( C^\infty \).

C. \textbf{If } r = \omega \textbf{ then the following conditions are equivalent:}

(1) \( v \) is finitely \( V \)-sufficient in \( C^\omega \),
(2) \( v \) is finitely \( C^0 \)-sufficient in \( C^\omega \),
(3) 0 is an isolated critical point of \( w \) (or it is not a critical one).

For the proof observe first that in cases A, C the implication (2) \( \Rightarrow \) (1)
is trivial, as well as the implications (5) \( \Rightarrow \) (4) \( \Rightarrow \) (1), (5) \( \Rightarrow \) (2) \( \Rightarrow \) (1) in
case B. Next, (3) \( \Rightarrow \) (2) in cases A, C and (3) \( \Rightarrow \) (5) in case B are con-
sequences of the Kuiper-Kuo theorem. Thus it is sufficient to prove (1) \( \Rightarrow \) (3)
in all three cases.

Observe now that in each case the condition (1) implies that for any \( C^r \)
realization \( \tilde{w} \) of \( v \) (resp. of \( v_0 \) in case C) the set \( \tilde{w}^{-1}(0) \) must be a
topological manifold of codimension 1 in a neighbourhood of 0 except 0. This
follows from the fact that for some \( c_i \) the value 0 is a regular value for the
function \( x \to \tilde{w}(x) + \Sigma c_i \lambda(x_i) \) restricted to a punctured neighbourhood of 0; where
we take \( \lambda : \mathbb{R} \to \mathbb{R} \) of class \( C^\infty \), flat at 0, with no other zeros, for cases A,
B; and we take \( \lambda(t) = t^N \) with \( N > s \) for case C. This is by an argument of
Thom ** which can be formulated as follows.

Let \( F : G \times H \to \mathbb{R}^{k+l} \) \((G, H \text{ open in } \mathbb{R}^k, \mathbb{R}^l) \) be a \( C^\infty \) map. If \( c \) is a regular
value of \( F \), then it is also of the map \( G \ni x \to F(x, y) \in \mathbb{R}^m \) for almost every

\* This is equivalent to the ellipticity of the ideal \( \left( \frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_n} \right) \).

\** Communicated by J. Mather.