§1. The results.

Denote by $K$ a complete field of characteristic zero with non-trivial valuation and let $A$ be the ring of germs at $0 \in K^n$ of analytic functions of $n$ variables (we say that a $K$-valued function defined in an open subset of $K^n$ is analytic if it is locally developable as a power series with coefficients in $K$ which has a non-zero radius of convergence with respect to the given valuation).

We say that a polynomial of degree $< r$ (i.e., $r$-jet) $w: K^n \to K$ is analytically sufficient if for any analytic function $f = \sum_{|\alpha| \leq r} a_\alpha x^\alpha$, defined in a neighbourhood of $0 \in K^n(a = (a_1, \ldots, a_n))$, $x = (x_1, \ldots, x_n)$ such that the $r$-jet $j^{(r)}(f) = \sum_{|\alpha| \leq r} a_\alpha x^\alpha = w$, there exists a local analytic isomorphism $h$, with $h(0) = 0$, $|\alpha| \leq r$ such that $f \circ h = w$ in a neighbourhood of $0$.

An analytic function $f$ defined in a neighbourhood of $0 \in K^n$ (or a germ $f \in A$) is called analytically $r$-determined if its $r$-jet $j^{(r)}(f)$ is analytically sufficient; $f$ is analytically finitely determined if $f$ is analytically $r$-determined for some $r \in \mathbb{N}$.

As an immediate consequence of this definition, we get that an analytically finitely determined germ $f \in A$ is analytically equivalent to a germ of a polynomial.

The main theorems of this note are Theorem 2 and 3. The first of these, which has also been proved (in the case $K = \mathbb{R}$ or $\mathbb{C}$) by different methods by J. Mather [3] and J.-C. Tougeron [7], gives a characterization of finitely determined functions. The second is closely related with Thom's Bombay Theorem 4 [6].

For $f \in \mathcal{F}$, denote by $\mathcal{P}(f) = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$ the ideal of the ring $\mathcal{F} = K[[x]]$ of power series in $x = (x_1, \ldots, x_n)$ generated by the formal power
series $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ of partial derivatives of $f$. Let $\mathfrak{m}$ be the maximal ideal of $\mathfrak{I}$.

**Theorem 1.** Let $f \in \mathfrak{I}$; assume $\mathfrak{m}^3 \subseteq \mathcal{P}(f)$ (where $s \in \mathbb{N}$). Then the $2s$-jet of $f$ is analytically sufficient.

The proof of Theorem 1 will be given in §2. The method of proof is purely algebraic. First, given $g \in f + \mathfrak{m}^{2s+1}$, we construct by the standard method (see e.g. [5]) a "formal isomorphism" $\mathfrak{h}$, such that $f \circ \mathfrak{h} = g$. Next, by a theorem of M. Artin [2], we can replace the formal solution by an analytic one. This argument will give somewhat more than we have stated. In fact we can choose an analytic solution $h$ such that in a neighbourhood of $0 \in k^n$ $h(x) = x + \hat{h}(x)$, where $\hat{h}_i \in \mathfrak{m}^{s+1}$ ($i = 1, \ldots, n$).

Let $\Omega(s) = \{a = (a_1, \ldots, a_n) : 0 \leq |a| \leq s\}$ and for $f \in \mathfrak{I}$ form the matrix

$$A(f,s) = (u^\beta_{(i,a)}) (i,a) \in \Omega(s), \beta \in \Omega(s),$$

where $u^\beta_{(i,a)}$ denotes the coefficient of the monomial $x^a$ in the series $x^\beta \frac{\partial f}{\partial x_1}$.

The proof of the following proposition will be given in §3; we shall use some arguments which are essentially due to Tougeron [7].

**Proposition 1.** (a) \(\dim \mathcal{P}(f) < s \Rightarrow \mathfrak{m}^s \subseteq \mathcal{P}(f)\)

(b) \(\mathfrak{m}^{s+1} \subseteq \mathcal{P}(f) \Rightarrow \dim \mathcal{P}(f) < \dim \mathcal{P}/\mathfrak{m}^{s+1} = \binom{n+s}{n}\)

(c) \(\text{rank } A(f,s) < \binom{n+s}{n} - s \Leftrightarrow \dim \mathcal{P}/\mathcal{P}(f) > s.\)

**Corollary 1.** If, for $f \in \mathfrak{I}$ and $s \in \mathbb{N}$, rank $A(f,s) > \binom{n+s}{n} - s$ then the $2s$-jet of $f$ is analytically sufficient.

**Example 1.** Let $f(x_1, x_2) = \sum_{i,j=0}^{s} a_{ij} x_1^i x_2^j$ be an analytic function of two variables ($n = 2$). If $a_{01}$ or $a_{10}$ is nonzero, then $\mathcal{P}(f) = \mathfrak{I}$ and $f$ is 1-determined (the results do not apply for $s = 0$).

If $a_{01} = a_{10} = 0$, $A(f,1)$ has rank $2 > 3 - 1$ if and only if $a_{11}^2 \neq 4a_{20}a_{02}$; when this holds, $f$ is 2-determined. If it does not, consider the matrix $A(f,2)$: