Transfer, symmetric groups, and stable homotopy theory*

Stewart B. Priddy

In the earlier talks of Segal [S1] and Quillen [Q] we have seen that the K-theory of the category of finite sets and permutations together with the composition law of disjoint union corresponds to stable cohomotopy theory. This talk is intended as an exposition of this fact from a somewhat more group-theoretic point of view. Several applications and techniques are also discussed.

The first part of the talk (§1) is a detailed explanation of the connection between symmetric groups and the stable homotopy of spheres. For completeness I have included references to most of the literature on this subject. After Quillen's remarks on $K_*(\mathbb{Z})$, I hope this material will be of interest to K-theorists. In the second part (§2,3), this connection together with a stable geometric realization of the transfer are used to obtain some new results in stable homotopy theory. At this point one is led to study the transfer in the homology of symmetric groups and so the last part of the talk (§4) deals with computing the transfer.

§1 Symmetric groups and stable homotopy theory

We begin by showing that the symmetric groups $\mathscr{S}_n$ and the spaces $\Omega^n S^n$ of pointed maps $S^n \to S^n$ are related by a map

$$\varphi : \mathbb{Z} \times B\mathscr{S}_\infty \to \Omega^\infty S^\infty$$

where $\mathscr{S}_\infty = \varinjlim \mathscr{S}_n$, $\Omega^\infty S^\infty = \varinjlim \Omega^n S^n$. This map is of interest because it provides a sort of homology approximation for a space whose homotopy groups are the stable homotopy groups of spheres, that is

*This talk is based on joint work with Daniel S. Kahn.

$$\varphi_* \; : \; H_*(\mathbb{Z} \times B\,\mathscr{S}_\infty) \overset{\approx}{\to} H_*(\Omega^\infty S^\infty)$$

and

$$\pi_*\Omega^\infty S^\infty = \pi_*^s S^0.$$

Thus the principal idea of this approach is to use φ to bring the rich structure of symmetric groups to bear on stable homotopy theory.

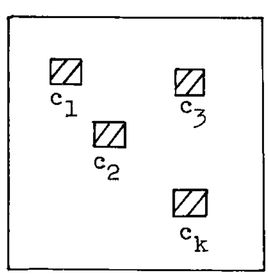
Now before describing φ we note that while $\mathscr{S}_\infty$ is not a finite group nevertheless the inclusion $\mathscr{S}_n \hookrightarrow \mathscr{S}_\infty$ induces an injection

$$H_*B\,\mathscr{S}_n \to H_*B\,\mathscr{S}_\infty$$

which is bijective for $* < \frac{n+1}{2}$. This fact is due to Nakaoka [N3]. The computation of the mod-p homology of the symmetric groups was completed by Nakaoka in [N4], culminating a remarkable series of papers [N1-4] on the subject.

<u>Construction of φ:</u> To describe φ we use Boardman's space $\mathscr{C}_n(k)$ of little "n-cubes" [B] (also see May [M1])

$$\mathscr{C}_n(k) = \{(c_1,\ldots,c_k): c_i = J_{i_1}\times\ldots\times J_{i_n},\ J_{i_j} = [a_{i_j}, b_{i_j}],$$
$$0 \le a_{i_j} < b_{i_j} \le 1;\ \text{int } c_i \cap \text{int } c_j = \emptyset,\ i \ne j\}$$



I^n

We topologize $\mathscr{C}_n(k)$ as a subset of $(I^{2n})^k$. Then $\mathscr{C}_n(k)$ is $(n-2)$-connected and $\mathscr{S}_k$ acts freely by permuting the c_i's. Now let $X = \Omega^n Y$ considered as the space of maps $(I^n, \dot{I}^n) \to (Y,*)$. Define a map