THE FRACTIONAL DERIVATIVE AND ENTIRE FUNCTIONS

MARVIN C. GAER AND LEE A. RUBEL

Abstract: For a class of functions defined on the real line, a fractional derivative and integral is defined which is an entire function of exponential type of the order. For simplicity, these operations will be called simply fractional differentiation. Properties of this derivative and its relation to other theories is studied.

Notation: The notation used will be explained in the body of this paper.

PREFACE

This paper is an expanded version of our earlier paper [1]. Although many of the main results appear there, most of the details and applications were omitted. In particular Chapter 3 was not included, and the results following from Buck's interpolation theorem, given at the end of Chapter 1, are also new. Parts of this work appeared in the first author's doctoral dissertation [9]. This dissertation also contains a fairly complete bibliography of over one hundred and seventy entries. The research of the first author was partially supported by a grant from the University of Delaware Research Foundation. The research of the second author was partially supported by a grant from the United States Air Force Office of Scientific Research, Grant No. AFOSR 68 1499.

* * *

INTRODUCTION

Let G be the class of functions, defined and analytic on a neighborhood of the extended real axis in the complex plane, that vanish at infinity. We show that for each $f \in G$ there exists a unique entire function $F$ of exponential type, whose rate of growth along the imaginary axis is less than $\pi$, such that $F(n) = \frac{f^{(n)}(0)}{n!}$. This leads us to define the derivative $f^{(z)}(0)$ of any complex order $z \neq -1, -2, -3, \ldots$ by $f^{(z)}(0) = \Gamma(z+1)F(z)$. This leads to a consistent theory of fractional differentiation and integration for functions in the class $G$. The existence of the
function $F$ is demonstrated by contour integration. That $F$ exists is related to the theorem of Leau, Faber, and Wigert [7, pp. 337-340] and to the work of Buck [4]. The uniqueness of $F$ is guaranteed by a theorem of Carlson. We show that our fractional integral coincides with the Weyl fractional integral for this class of functions over a suitable range of $z$, and also coincides with the result of formal differentiation under the integral sign in the Fourier transform representation, again for a suitable range of $z$. An application of contour integration reminiscent of the Paley-Wiener circle of ideas characterizes the Fourier transforms of functions in the class $G$ as the restrictions of two entire functions of exponential type, one decaying exponentially along the positive real axis, and the other decaying exponentially along the negative real axis.

A new formula for the fractional derivative of a product is derived that is not a generalization of the Leibnitz formula, and will not reduce to it, even in the case of first order differentiation. Finally, we obtain some uniqueness and parity results for the class $G$. We show, for instance, that if $f \in G$, and if at two points of the real axis a sufficient number of the derivatives of $f$ behave as though $f$ were an even or odd function, then $f$ must actually be even or odd.

CHAPTER 1. OUR DEFINITION OF THE FRACTIONAL DERIVATIVE.

In this chapter we introduce the class $G$ of functions, define the fractional derivative for such functions, and develop the fundamental theory.

Definition 1.1. Let $G$ denote the class of complex functions which are analytic in an open set containing the real line and the point at infinity and which vanish at infinity.

Thus, $G$ is the class of functions vanishing at $\infty$ which are regular everywhere except in one or two bounded "patches", one in the upper half-plane and one in the lower half-plane, not intersecting the real axis. Since the functions in $G$ are regular and vanishing at $\infty$, they can be expanded near $\infty$ in series of powers of $1/z$, with zero constant term. These functions thus are of the