FRACTIONAL SPACES OF TEMPERATE DISTRIBUTION

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Abstract: The use of fractional derivatives enables a natural construction of a family of Hilbert spaces $H^\lambda$, $\lambda \geq 0$, such that the union of all $H^\lambda$ is the space $\mathcal{S}'(\mathbb{R}^n)$ of temperate distributions. For each $\lambda \geq 0$ the largest locally convex space $\mathcal{O}_\lambda$ of functions, by which distributions from $H^\lambda$ can be sensibly multiplied, is defined and the continuity of multiplication on $\mathcal{O}_\lambda \times H^\lambda$ is established.

In [3] and [4], we defined Hilbert spaces

$$H^k(\mathbb{R}^n) = \{ f \in L^2(\mathbb{R}^n) ; \sum_{|\alpha + \beta| \leq k} \int_{\mathbb{R}^n} |x^\alpha D^\beta f(x)|^2 dx < \infty \} , \quad k \in \mathbb{N} ,$$

with an inner product

$$(f,g)_k = \sum_{|\alpha + \beta| \leq k} (2\pi)^2 |\alpha| \int_{\mathbb{R}^n} x^\alpha D^\beta f(x) \overline{x^\alpha D^\beta g(x)} \, dx ,$$

where $N = \{0,1,2,\ldots\}$, $\alpha \in \mathbb{N}^n$, $|\alpha| = \sum_{j=1}^{\infty} \alpha_j$ , $D^\alpha f = \frac{\partial^{\alpha}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} f$ , $x^\alpha = \prod_{j=1}^{\alpha_j} x^j$ , and proved that the intersection $\bigcap_{k \in \mathbb{N}} H^k$, equipped with the initial topology, see [2], is the locally convex space $\mathcal{S}(\mathbb{R}^n)$, so called space of rapidly decreasing functions, with a topology defined by seminorms

$$(1) \quad q_{a,b}(f) = \sup_{\xi \in \mathbb{R}^n} |x^{\alpha} D^\beta f(x)| , \quad a,b \in \mathbb{N}^n .$$

Let $H^{-k}$ be the strong dual of $H^k$ and $\mathcal{S}'(\mathbb{R}^n)$ the dual of $\mathcal{S}(\mathbb{R}^n)$ . Throughout this paper $\mathcal{S}'$ has the strong topology $\mathcal{B}(\mathcal{S}',\mathcal{S})$ .

Then the set of all functionals from $\bigcup_{k \in \mathbb{N}} H^{-k}$ restricted to $\mathcal{S}(\mathbb{R}^n)$ equals $\mathcal{S}'(\mathbb{R}^n)$ and the final locally convex topology of $\bigcup_{k \in \mathbb{N}} H^{-k}$, see [2] and [5], equals $\mathcal{B}(\mathcal{S}',\mathcal{S})$ . The Fourier transformation, with the kernel $(-2\pi ix,\xi)$, is a unitary mapping on each $H^k$, $k = 0, \pm 1, \pm 2, \ldots$.
In the first part of this paper we define Hilbert spaces $H^K$ for any real $K$ so that above mentioned properties are preserved. In the second part we construct the largest space, which we denote by $O_K$, where $K \geq 0$, of functions such that for any $u \in O_K$ the mapping $(u,f) \mapsto uf$ from $H^K$ into $S'$ makes sense and is continuous. Finally, we provide $O_K$ with a natural topology under which the multiplication is continuous on $O_K \times H^K$ and the intersection $\bigcap_{K \geq 0} O_K$ with the initial topology equals (topologically) the space $O_0$ of multipliers.

Notation. $R^n$ is the Euclidean $n$-space with an inner product $(\cdot,\cdot)$ and $R^n_{++} = (0,\infty)^n$ its first orthant. For $\alpha \in R^n$ we write $[\alpha] = ([a_1],[a_2],\ldots,[a_n])$, where $[a_j]$ is an integer satisfying inequalities $[a_j] \leq a_j < [a_j]+1$. $L^2(R^n)$ is the Hilbert space of all square integrable functions with a norm $\|\cdot\|$, $L_{\text{loc}}(R^n)$ is the space of all functions locally integrable on $R^n$, $C^\infty(R^n)$ is the space of all functions infinitely differentiable on $R^n$ which have compact support. We do not define any topology on either $L_{\text{loc}}(R^n)$ or $C^\infty(R^n)$. It is convenient to introduce a weight function $W(x) = (1+(x,x))^{\frac{1}{4}}$, $x \in R^n$, and a multi-index $\omega = (1,1,\ldots,1) \in N^n$.

The Fourier, resp. the inverse Fourier, transform of a distribution $f \in S'$ is denoted by $Ff$ or $\hat{f}$, resp. $F^{-1}f$ or $\check{f}$. We say that $f \in L_{\text{loc}}(R^n)$ has a generalized, or Sobolev, $\alpha$-derivative $g$ if
\[ \int_{R^n} g \phi \, dx = (-1)^{[\alpha]} \int_{R^n} f \phi^{\alpha} \, dx \] holds for each $\phi \in C^\infty_0(R^n)$. We write $g = D^\alpha f$.

Let $f \in L_{\text{loc}}(R^n)$, $\alpha \in R^n_{++}$, $\beta = [\alpha]-\alpha$, and $g(x) = \left( \prod_{k=1}^{n} \Gamma((1+\beta_k))^{-1} ight) \prod_{k=1}^{n} \frac{x_k^{\beta_k}}{\Gamma(\beta_k+1)^{-1}} \int_{R^n} f(t) \prod_{k=1}^{n} (x-t_k)^{\beta_k} \, dt$, where $\Gamma$ is the Euler gamma function. If the (classical) derivative $D^{[\alpha+\omega]}g$ exists it is called the fractional $\alpha$-derivative of $f$ and denoted by $D^\alpha f$. If $W^{[\omega]}f \in L^2(R^n)$ then the generalized derivative $D^{[\alpha+\omega]}g$ exists and equals $F^{-1}((2\pi i \alpha)^{\omega} \hat{f})$, where $(2\pi i \alpha)^{\omega} = \prod_{k=1}^{n} (2\pi i x_k)^{\omega_k} \exp(\sum_{k=1}^{n} \frac{\alpha_k}{2\pi i x_k}(2-\text{sgn}(x_k)))$. For $\alpha \notin N^n$ the power $(2\pi i)^{\alpha}$ is not uniquely defined. Hence the derivative $D^\alpha f$ is not uniquely defined either. This inconvenience is however outweighed by the ease with which $F^{-1}((2\pi i)^{\omega} \hat{f})$ can be handled in calculations.

Auxiliaries. Lemma 1. For each $f \in L^2(R^n)$ there exists a sequence $\phi_s \in C^\infty_0(R^n)$, $s \in N$, such that $D^\alpha \phi_s \to D^\alpha f$ in the topology of $L^2$ for any $\alpha \in N^n$ for which the generalized $D^\alpha f$ exists and belongs to $L^2(R^n)$.