APPLICATIONS OF FRACTIONAL CALCULUS TO SPHERICAL (RADIAL) PROBABILITY MODELS AND GENERALIZATIONS

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Abstract: In the statistical study of radial (known also as spherical) probability models, conditional distributions ("Thompson distributions") on the sphere and minimum variance unbiased estimation involve fractional calculus. Projections of spherical distributions on lower dimension spaces, that is, distributions of projections of spherically distributed vectors and their "anti-projections"—which generalize univariate symmetric models to multivariate models—are obtained by fractional integration and differentiation. A calculus of operators based on the group of Fourier-Hankel type transforms and fractional integration-differentiation to apply to probability densities of spherical models is outlined. The problem of comparing spherical models to the normal Gaussian model through series expansion is touched upon. A matric generalization of fractional calculus, motivated by the study of the family of hyperspherical distributions of random matrices, is suggested for further research and consideration.

Notation: Fractional integration will be defined as a Weyl transform:
\[ D^{-v}_+ f(x) = \frac{1}{\Gamma(v)} \int_x^{+\infty} (t-x)^{v-1} f(t) dt, \quad v > 0, \]

fractional differentiation as:
\[ D^v_+ f(x) = \frac{d^m}{dx^m} \left( \frac{(-1)^m}{\Gamma(p)} \int_x^{+\infty} (t-x)^{p-1} f(t) dt, \quad v > 0, \quad 0 < p < m - v < 1. \]

The definition
\[ 0D^v_+ f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t) dt, \quad v > 0, \]
will also be used once. When no confusion is possible, the letter D will be used without subscript.

To avoid a heavy display of symbols, probability density may, occasionally in the text, be defined up to a multiplicative constant and characteristic function and Fourier transform as well, as their normalization is straightforward (proportionality will be denoted by the symbol $\sim$); e.g., characteristic function $\phi(u) \sim h(u)$ implies $\phi(u) = h(u)/h(0)$ and probability density $f(x) \sim w(x)$ implies $f(x) = w(x) / \int_{-\infty}^{+\infty} w(x) dx$.

The reader should also keep in mind that the definitions of the Fourier transforms found in the literature may or may not involve normalizing or scale factors in $\pi$ and that the definition of the Hankel transform varies from author to author (the different definitions are straightforward to reconcile).

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1. Thompson's distributions and fractional calculus.

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random vector whose numerical values $\mathbf{X} = (x_1, x_2, \ldots, x_n)$ are distributed with probability density function (p.d.f.) $f_n(\mathbf{X}) = g_n(\mathbf{X}' / \theta)$, where $\mathbf{X}' = \mathbf{X}' / \theta$. One says that $\mathbf{X}$ has a radial or a spherical distribution. The use by the statisticians of the word spherical rather than radial stems from the fact that the degenerate conditional distribution of $\mathbf{X}$, given $|\mathbf{X}| = r$ is uniform on the sphere of radius $r$. The p.d.f. is invariant under the group of orthogonal transformations of $\mathbf{X}$, in particular, under the group of rotations about the origin. The distribution admits $\mathbf{R}^2 = \mathbf{X}'$ as a sufficient statistic, namely, the conditional distributions, given $\mathbf{R}^2 = r^2$, of all statistics, functions of $\mathbf{X}$, are independent from $\theta$. Also $\mathbf{R}^2$ is an "absolutely sufficient" statistic, that is, these distributions do not depend on which specific distribution $g_n$ is considered but only on the sphericity of the class. $\mathbf{X}/\mathbf{R}$ and $\mathbf{R}$ are independent random variables and the conditional and marginal distribution of the ancillary statistic $\mathbf{X}/\mathbf{R}$ is uniform on the unit sphere. A special case of this model is when $\mathbf{X}$ is a random sample of $n$ observations of a random variable $X$ whose distribution is Laplace-Gaussian (the so-called normal distribution) with mean zero and variance $\theta$. It is known that, under mild conditions, the