FRACTIONAL INTEGRATION OF FUNDAMENTAL SOLUTIONS

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Abstract: This paper concerns the question, "How can we find a fundamental solution of

\[ U_{xx} + U_{yy} + U_{zz} + w^2 c^{-2}(z) U = 0 \]

when we know a fundamental solution of

\[ U_{xx} + U_{zz} + w^2 c^{-2}(z) V = 0 ? " \]

This question is answered by using fractional integration and the example \( c(z) = z \) is examined.

A similar question, answer, and example are presented in three papers by Erdélyi, [1] to [3]. These are discussed in this paper and found to be inadequate for our purposes because his technique does not transform fundamental solutions into fundamental solutions, but rather into another type of solution.

Notation:

\[ K^\alpha_{-n} f(r) = \frac{n}{r^{(n)}} \int_r^\infty (t^n - r^n)^{\alpha-1} f(t) t^{n-1} dt. \]

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HADAMARD'S METHOD OF DESCENT

If we think of a fundamental solution as being the solution

\[ U_{xx} + U_{yy} + U_{zz} + w^2 c^{-2}(z) U = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0), \]

it seems reasonable to expect that if we integrate each term of this equation with respect to \( y \) between the limits of plus and minus infinity, and invert the order of integration and differentiation, that we would obtain a fundamental solution satisfying

\[ V_{xx} + V_{zz} + w^2 c^{-2}(z) V = \delta(x-x_0)\delta(z-z_0). \]

This is Hadamard’s method of descent [5].
Since the coefficient \( c(z) \) depends on only \( z \), it is well known that we may take \( x_0 = y_0 = 0 \) and that the fundamental solution \( U \) has to have the form \( U(\sqrt{x^2 + y^2}, z, z_0) \). Hadamard’s method of descent gives

1. \( V(x, z, z_0) = 2 \int_0^\infty U(\sqrt{x^2 + x^2}, z, z_0) \, dx \).

With the change of variable \( t^2 = s^2 + x^2 \), we obtain

\[
V = \int_X \frac{U(t, x) \, 2t \, dt}{\sqrt{t^2 - x^2}}.
\]

That is,

2. \( V = \sqrt{\pi} \, K_{\frac{1}{2}} U(x, z) \) or \( U = \frac{1}{\sqrt{\pi}} \, K_{-\frac{1}{2}} V(x, z) \)

where

3. \( K^{\alpha}_{r^n} f(r) = \frac{n}{\Gamma(\alpha)} \int_\Gamma (t^n - r^n)^{\alpha-1} f(t) t^{n-1} dt \)

is the standard notation \([1], \text{Eq. 3.3}\) for the Weyl fractional integral with respect to \( r^n \). Aside from the definition in (Eq. 3), the only facts about fractional integrals we use are the exponent laws \([1], \text{Eq. 3.1} \text{ and Eq. 3.4}\):

4. \( K^{\alpha}_{r^n} K^{\beta}_{r^n} = K^{\alpha+\beta}_{r^n} \)

and

5. \( K^0_{r^n} f(r) = f(r) \).

It is useful to notice that

6. \( K^{\alpha}_{x^2} f(x) = [K^{\alpha}_x f(\sqrt{x})]_{x=x^2} \)

and that

7. \( K^{\alpha}_{x^2} f(x+a) = [K^{\alpha}_x f(x)]_{x=x+a} \).

An example will show how this method is used. For the case \( c = 1 \), we have \( V = H^{(1)}_0 (w\sqrt{x^2 + z^2}) \). We will now deduce the known result \( U = e^{-i\omega p^2 + z^2} / (p^2 + z^2) \) where \( p^2 = x^2 + y^2 \) by the above method. From (Eq. 2) and the exponent law, we find