ON THE RECENT TRENDS IN THE DEVELOPMENT, THEORY
AND APPLICATIONS OF FRACTIONAL CALCULUS

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Abstract: As is well-known, there is a number of possibilities for the solution of the fundamental problem of fractional (integro-differential) calculus: "find the simplest common generalization of the derivation and integration processes by means of interpolation relating to the index (order) of the mentioned operations". After a brief discussion of the main directions in the development of the theory, a survey of the corresponding application topics is given (theory of functions, integral transformations, theory of approximations and summability, differential and integral equations, operator theory, generalized differentiation of discontinuous functions), particular stress being laid upon some results of the last decades.

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It is well-known that integrals and derivatives of fractional order were properly introduced in analysis by Abel, Liouville, and Riemann in the first half of the nineteenth century, but the use of generalized differential and integral operators has become somewhat more familiar only since the last decades of the nineteenth century by the symbolic calculus of Heaviside, and furthered by the works of such mathematicians as Hadamard, Hardy and Littlewood, M. Riesz, and H. Weyl. The main purpose of this lecture is to review certain investigations published in the field during the last decades; with this end in view, we first have to consider the fundamental problem of the theory: "find the simplest common generalization of the derivation and integration processes by means of interpolation relating to the index (order) of the mentioned operations". This fundamental problem can be approached in various ways.

If we start with a classical formula of Cauchy for the m-th iterated integral of a function $f$ continuous over $[x_0, x]$ which yields the unique solution of the initial value problem

$$y^{(m)}(x) = f(x); \ y(x_0) = y'(x_0) \ldots = y^{(m-1)}(x_0) = 0$$
and replace $m$ by a continuous parameter $v > 0$ as well as $(m-1)!$
correspondingly by $\Gamma(v)$, we obtain the usual definition of the frac­
tional integral of order $v$, due to Liouville and Riemann:

$$
(1) \quad x_0^v I_x^v f = \frac{1}{\Gamma(v)} \int_{x_0}^{x} f(t)(x-t)^{v-1} dt.
$$

Of course, the condition on $f$ may be weakened: in case of any
Lebesgue integrable function and any fixed $v > 0$, the existence of
the integral (1) is assured for almost all $x$; if $f$ is also
bounded in $[x_0, x]$, then (1) exists for every $v > 0$, furthermore the
operator $x_0^v I_x$ satisfies the so-called index law (or semigroup prop­
erty):

$$
(2) \quad x_0^v I_x^v \left( x_0^u I_x^u f \right) = x_0^{v+u} I_x^v f \left( v_1 > 0, u_2 > 0; x_0 < t \leq x \right).
$$

Note that all this holds for complex $v$ with $\text{Re } v > 0$, too; the Riemann­
Liouville integral (1) is then an analytic function of $v$.

The extension of (1) for negative values of $v$, i.e. the
introduction of a fractional derivative

$$
(3) \quad x_0^{-v} D_x^{-v} f = x_0^v I_x^v f \quad (v < 0)
$$

may be obtained simply by ordinary differentiation of fractional inte­
grals, namely by Riemann's formula [1]

$$
(4) \quad x_0^{-p} D_x^{-p} f = \frac{d^m}{dx^m} x_0^{m-p} f \quad (m \equiv 0),
$$

where $m$ denotes the least integer greater $m$. Unfortunately, (4)
is not a "right" generalization of the ordinary derivatives $f^{(p)}(x)$
($p=0, 1, 2, \ldots$), because the mere existence of $f^{(p)}(x)$ does not imply
the relation $x_0^p D_x^p f = f^{(p)}(x)$; for this result, we need also a supple­
mentary restriction: the continuity of $f^{(p)}$ at the point $x$. Thus
there are many problems, for which the definition (4) of the fraction­
al derivative is inadequate and it must be replaced by a better one.