EXPONENTIAL REPRESENTATION OF SOLUTIONS OF ORDINARY
DIFFERENTIAL EQUATIONS

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I shall describe here a kind of calculus for solutions of ordinary differential equations developed jointly with my collaborator A. Agrachev. This calculus is based on the exponential representation of the solutions and reflects their most general group-theoretic properties. In deriving the calculus we were strongly influenced by problems of control and optimization and it is shaped according to the needs of these theories. Nevertheless it might be considered, as I believe, not merely as a technical tool for dealing with control problems only but could also be of more general interest. This may justify my choice of the topic for the Equadiff conference.

1. Differential equations considered

Let us consider a differential equation in $\mathbb{R}^n$

\[ z = X_t(z) \]

where $X_t(z)$ is a $C^\infty$-function in $z \in \mathbb{R}^n$ for $\forall t \in \mathbb{R}$, measurable in $t$ for $\forall z \in \mathbb{R}^n$ and satisfying the condition

\[ \| X_t \|_k \leq \mu_k(t), \quad \int_{\mathbb{R}} \mu_k(t) dt < \infty, \quad k = 0, 1, \ldots \]

where $\| \cdot \|_k$ denotes the norm of the uniform convergence in $\mathbb{R}^n$ up to the $k$-th derivative.

Our first goal is to find a suitable representation of the flow defined by (1), that is, of a family of $C^\infty$-diffeomorphisms $F_t$, $t \in \mathbb{R}$, of $\mathbb{R}^n$ satisfying the equation

\[ \frac{d}{dt} F_t x = X_t(F_t x), \quad F_0 = \text{Id} \quad \forall x \in \mathbb{R}^n. \]

The existence of $F_t$ is guaranteed by (2).

2. Transforming (3) into a linear "operator equation"

There is a procedure transforming the nonlinear equation (3) into a certain linear "operator equation" for $F_t$. To describe it let me introduce some standard notions.

$\mathcal{A}$ will denote the algebra of all $C^\infty$-scalar functions $f, g, \ldots$ on $\mathbb{R}^n$ with the topology of the uniform convergence on compact sets for every derivative. $\mathcal{A}$ stands for the associative algebra
of all continuous linear transformations of \( \Phi \). The composition of two elements \( A_1, A_2 \) in \( \mathcal{A} \) will be denoted by \( A_1 \circ A_2 \). The operators from \( \mathcal{A} \) can be applied also to vector-valued functions on \( \mathbb{R}^n \). Denote by \( \Theta \) the identity mapping of \( \mathbb{R}^n : \Theta(x) \equiv x \).

We shall say that a sequence of operators \( A_1, A_2, \ldots \) from \( \mathcal{A} \) is convergent to \( A \) iff the sequence of functions \( A_1 \Theta, A_2 \Theta, \ldots \) converges in \( \Phi \) to \( A \Theta \). Every diffeomorphism \( F \) of \( \mathbb{R}^n \) will be considered as an element of \( \mathcal{A} : Ff(x) = f(Fx), \forall x \in \mathbb{R}^n \), and the set of all \( C^\infty \)-diffeomorphisms of \( \mathbb{R}^n \) will be denoted by \( \mathcal{D} \).

By \( \mathcal{L} \) we shall denote the Lie algebra of all \( C^\infty \)-vector fields on \( \mathbb{R}^n \), which is a subspace of \( \mathcal{A} \) characterized by the differentiation rule \( X(fg) = (Xf)g + f(Xg), \forall X \in \mathcal{L}, \forall f, g \in \Phi \). The Lie bracket of two fields will be denoted as usual by \( [X,Y] = XoY - YoX = (ad X)Y \). The following important relation holds:

\[
(4) \quad F_0 X_0 F^{-1} \stackrel{\text{def}}{=} (Ad F)X \in \mathcal{L} \forall X \in \mathcal{L}, \forall F \in \mathcal{D}.
\]

Consider \( X_t, t \in \mathbb{R} \), in (1) as a nonstationary (time-dependent) vector field on \( \mathbb{R}^n \). It is not difficult to show that (3) is equivalent to the linear "operator equation" for the flow \( F_t \)

\[
(5) \quad \frac{d}{dt} F_t = F_t \circ X_t, \quad F_0 = Id \iff F_t = Id + \int_0^t F_\tau \circ X_\tau d\tau,
\]

where the operations of differentiation and integration in \( t \) should be understood in the "weak" sense: first apply the operator to an arbitrary function from \( \Phi \) and then differentiate or integrate. The equivalence between (3) and (5) should be understood literally - the existence of a unique solution of (3) implies the existence of a unique solution \( F_t \), \( t \in \mathbb{R} \), for (5) which at the same time necessarily turns out to be a flow and vice versa. Certainly we can always consider the flow \( F_t \) only for values of \( t \) sufficiently close to zero since the equation \( F_t = F_{t_0} + \int_{t_0}^t F_\tau \circ X_\tau d\tau \), \( t_0 \)-arbitrary fixed, has exactly the same properties as (5), which permits to restore the whole flow \( F_t \), \( t \in \mathbb{R} \).

Call the formal series

\[
(6) \quad Id + \sum_{i=1}^{\infty} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{i-1}} d\tau_i X_{\tau_1} \circ X_{\tau_2} \circ \cdots \circ X_{\tau_i},
\]

arising when solving the linear equation (5) formally, the Volterra