THE RAYLEIGH AND VAN DER POL WAVE EQUATIONS, SOME GENERALIZATIONS

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Here are two interesting nonlinear partial differential equations. The first we call the Rayleigh wave equation,

\[ y_{tt} - y_{xx} = \varepsilon(y_t - y_t^3) \]  \hspace{1cm} (1.1)
\[ y(t,0) = y(t,\pi) = 0, \]

and the second is the wave equation of Van der Pol type,

\[ y_{tt} - y_{xx} = \varepsilon(1 - y^2)y_t \]  \hspace{1cm} (1.2)
\[ y(t,0) = y(t,\pi) = 0 \]

Each of these has been used to model physical phenomena, although they first appeared in the literature as curiosities. For example, in [3] we see (1.1) serving as a model for the large amplitude vibrations of wind-blown, ice-laden power transmission lines. Equation (1.2), on the other hand, can describe plane electromagnetic waves propagating between two parallel planes in a region where the conductivity varies quadratically with the electric field [5].

Just as their counterparts from ordinary differential equations can be transformed one to the other, (1.1) and (1.2) are related. As we shall see, solutions to each can be obtained by simple operations performed on the solution of a certain first order, nonlinear wave equation. In fact, the goal of this note is to show how certain aspects of this particular equation can be studied such as global existence, uniqueness, and the transient and steady state behavior for small \( \varepsilon > 0 \).

It is perhaps surprising that some second order equations can be solved as first order problems. However, this is strongly suggested by the form of (1.1) where \( y \) itself is absent. Also, although two independent initial conditions are required for (1.1) and (1.2), each must be an odd, 2\( \pi \)-periodic function of \( x \). Two such odd functions can always be generated from an arbitrary 2\( \pi \)-periodic function by separating it into its odd and even parts and integrating or differentiating the latter. Obviously, this

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procedure can be reversed, and a single periodic function can be built from two odd functions. Thus it is possible for the initial value of an appropriate first order equation to carry the initial position and velocity for (1.1) and (1.2) in its odd and even parts.

Let us now derive the first order wave equation corresponding to (1.1). Let

\[ u = y_t - y_x \]  (1.3)

and let \( P \) project a periodic function of \( x \) to its odd part. Since \( y \) and \( y_t \) are odd in \( x \), \( y_t = Pu \). Hence, from (1.3),

\[ u_t + u_x = y_{tt} - y_{xx} = \varepsilon(y_t - y_t^3) = \varepsilon(Pu - (Pu)^3) \]  (1.4)

Similarly, we can obtain (1.4) from (1.2) by the transformation,

\[ y = \sqrt{3}z, \quad u = z - \int_0^x f \]  (1.5)

Of course, these derivations are formal, but they strongly suggest that once we have a solution to (1.4), then \( \sqrt{3} Pu \) will solve (1.1) and \( \int^t Pu \) will be a solution to (1.1). We shall not discuss here the question of whether these equations are equivalent. Rather we simply regard (1.4), or rather

\[ u_t + u_x = \varepsilon(Pu - h(Pu) + f(t,x)), \]  (1.6)

where \( h \) is a suitable monotone increasing function and \( f \) is a \( 2\pi \)-periodic forcing term depending on \( t \) and \( x \), as the fundamental equation generalizing the previous examples. We mention, however, that the equivalency of (1.4) and (1.1) has been established in [7].

Some names associated with the study of (1.1) and (1.2) are Kurzweil [8], [9], Vejvoda and Štěrý [10], Chikwendu and Kevorkian [2], and Fink, Hall, and Hausrath [4], [5], [7]. Kurzweil's contribution is by far the most important, as he was able to prove the existence of exponentially asymptotically stable integral manifolds of periodic solutions for both (1.1) and a general form of (1.2). Vejvoda and Štěrý showed the existence of periodic solutions to these equations by elementary methods. In [2], a formal analysis of (1.1) appears, using a two-time method from the theory of ordinary differential equations. In [4] a convergent two-time method is developed, and in [7] a rather detailed analysis is made of the Rayleigh equation.

Let us begin the analysis of (1.6) by studying the question of global existence and uniqueness.