11. Geometric realization of simplicial spaces

We shall use the technique of geometric realization of simplicial spaces to transfer the categorical constructions of the previous sections into constructions of topological spaces. This technique is an exceedingly natural one and has long been implicitly used in classifying space constructions. Segal [26] appears to have been the first to make the use of this procedure explicit.

In this section and the next, we shall prove a variety of statements to the effect that geometric realization preserves structure; thus we prove here that realization preserves cell structure, products (hence homotopies, groups, etc.), connectivity, and weak homotopy equivalences. Base-points are irrelevant in this section, hence we shall work in the category \( \mathcal{U} \) of compactly generated Hausdorff spaces.

Let \( \Delta_q \) denote the standard topological \( q \)-simplex,
\[
\Delta_q = \{ (t_0, \ldots, t_q) | 0 \leq t_i \leq 1, \sum t_i = 1 \} \subset \mathbb{R}^{q+1}.
\]
Define \( \delta_i : \Delta_{q-1} \to \Delta_q \) and \( \sigma_i : \Delta_{q+1} \to \Delta_q \) for \( 0 \leq i \leq q \) by
\[
\delta_i(t_0, \ldots, t_{q-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{q-1})
\]
and
\[
\sigma_i(t_0, \ldots, t_{q+1}) = (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{q+1})
\]
Definition 11.1. Let $X \in \Delta \mathcal{U}$. Define the geometric realization of $X$, denoted $|X|$, as follows. Let $\overline{X} = \sum_{q \geq 0} X_q \times \Delta_q$, where $X_q \times \Delta_q$ has the product topology (in $\mathcal{U}$) and $\sum$ denotes disjoint union. Define an equivalence relation $\sim$ on $\overline{X}$ by

$$(\delta_i x, u) \sim (x, \delta_i u) \quad \text{for } x \in X_q, \ u \in \Delta_{q-1},$$

$$(s_i x, u) \sim (x, \sigma_i u) \quad \text{for } x \in X_q, \ u \in \Delta_{q+1}.$$ 

As a set, $|X| = \overline{X}/(\sim)$. Let $F_q|X|$ denote the image of $\sum_{i=0}^q X_i \times \Delta_i$ in $|X|$ and give $F_q|X|$ the quotient topology. Then $F_q|X|$ is a closed subset of $F_{q+1}|X|$, and $|X|$ is given the topology of the union of the $F_q|X|$. The class of $(x, u) \in \overline{X}$ in $|X|$ will be denoted by $[x, u]$. If $f: X \rightarrow X'$ is a map in $\Delta \mathcal{U}$, define $|f|: |X| \rightarrow |X'|$ by $|f|[x, u] = [f(x), u]$. Observe that if each $f_q$ is an inclusion (resp., surjection), then $|f|$ is an inclusion (resp., surjection).

Of course, if $X$ is a simplicial set, then the classical geometric realization of $X$, due to Milnor, coincides with the geometric realization of $X$ regarded as a discrete simplicial space. Further, if $\overline{X}$ denotes the underlying simplicial set of a simplicial space $X$, then $|X| = |\overline{X}|$ as sets and therefore any argument concerning the set theoretical nature of $|\overline{X}|$ applies automatically to $|X|$. The following definition will aid in the analysis of the topological properties of $|X|$. 
