A FAST ALGORITHM FOR NONLINEARLY CONSTRAINED OPTIMIZATION CALCULATIONS

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1. Introduction

An algorithm for solving the general constrained optimization problem is presented that combines the advantages of variable metric methods for unconstrained optimization calculations with the fast convergence of Newton's method for solving nonlinear equations. It is based on the work of Biggs (1975) and Han (1975, 1976). The given method is very similar to the one suggested in Section 5 of Powell (1976). The main progress that has been made since that paper was written is that through calculation and analysis the understanding of that method has increased.

The purpose of the algorithm is to calculate the least value of a real function $F(\underline{x})$, where $\underline{x}$ is a vector of n real variables, subject to the constraints

$$\left.\begin{array}{l} c_i(\underline{x}) = 0, \quad i = 1, 2, \ldots, m' \\ c_i(\underline{x}) \geqslant 0, \; i = m'+1, m'+2, \ldots, m \end{array}\right\} \qquad (1.1)$$

on the value of $\underline{x}$. We suppose that the objective and constraint functions are differentiable and that first derivatives can be calculated. We let $\underline{g}(\underline{x})$ be the gradient vector

$$\underline{g} = \underline{\nabla}\, F(\underline{x}) \qquad (1.2)$$

and we let N be the matrix whose columns are the normals, $\underline{\nabla} c_i$, of the "active constraints".

The given algorithm is a "variable metric method for constrained optimization". The meaning of this term is explained in Section 2. Methods of this type require a positive definite matrix of dimension n to be revised as the calculation pro-

ceeds and they require some step-length controls to force convergence from poor starting approximations. Suitable techniques are described in Sections 3 and 4 and thus the recommended algorithm is defined. It is applied to some well-known examples in Section 5 and the numerical results are excellent. It seems to be usual for our algorithm to require less than half of the amount of work that is done by the best of the other published algorithms for constrained optimization. A theoretical analysis of some of the convergence properties of our method is reported elsewhere (Powell, 1977).

2. Variable metric methods for constrained optimization

Variable metric methods have been used successfully for many years for unconstrained optimization calculations. A good survey of their properties in the unconstrained case is given by Dennis and Moré (1977). Each iteration begins with a starting point $\underline{x}$ in the space of the variables at which the gradient (1.2) is calculated. A positive definite matrix, B say, which is often set to the unit matrix initially, defines the current metric. The vector $\underline{d}$ that minimizes the quadratic function

$$Q(\underline{d}) = F(\underline{x}) + \underline{d}^T \underline{g} + \tfrac{1}{2} \underline{d}^T B\underline{d} \tag{2.1}$$

is calculated. It is used as a search direction in the space of the variables, $\underline{x}$ being replaced by the vector

$$\underline{x}^* = \underline{x} + \alpha \underline{d}, \tag{2.2}$$

where α is a positive multiplier whose value depends on the form of the function of one variable

$$\phi(\alpha) = F(\underline{x} + \alpha \underline{d}). \tag{2.3}$$

The matrix B is revised, using the gradients $\underline{g}$ and $\underline{g}^* = \underline{\nabla} F(\underline{x}^*)$. Then another iteration is begun. We follow Han (1976) in seeking extensions to this method in order to take account of constraints on the variables.

When there are just n equality constraints on $\underline{x}$ and when the matrix of constraint normals

$$N = \{ \underline{\nabla} c_1, \underline{\nabla} c_2, \ldots, \underline{\nabla} c_{m'} \} \tag{2.4}$$