Chapter 4

QUANTUM PARTICLE SCATTERING IN POTENTIALS
POSITIVE POWERS OF THE DIRAC $\delta$ DISTRIBUTION

§1. INTRODUCTION

Potentials with strong local singularities have been studied in scattering theory, [3], [27], [28], [115], [116], [140]. The strongest local singularities of the potentials considered were those of measures which need not be absolutely continuous with respect to the Lebesgue measure, [27]. The potentials in this chapter, given by arbitrary positive powers $(\delta)^m$, $0 < m < \infty$, of the Dirac $\delta$ distribution, present obviously stronger local singularities.

The wave function solutions obtained possess the scattering property of being given by pairs $\psi_-, \psi_+$ of usual $C^\infty$ solutions of the potential free motions, each valid on the respective side of the potential and satisfying special junction relations on the support of the potentials. In the case of the potential $\alpha\delta$, i.e. $m = 1$, the only one treated in literature, [44], the junction relation obtained is identical with the known one.

§2. WAVE FUNCTIONS, JUNCTION RELATIONS

One and three dimensional motions are considered. The one dimensional wave function $\psi$ is given by

(1) $\psi''(x) + (k-U(x))\psi(x) = 0$, $x \in \mathbb{R}^1$ ($k \in \mathbb{R}^1$)

with the potential

(2) $U(x) = \alpha(\delta(x))^m$, $x \in \mathbb{R}^1$ ($\alpha \in \mathbb{R}^1$, $m \in (0,\infty)$)

The solution of (1), (2) is expected to be of the form

(3) $\psi(x) = \begin{cases} \psi_-(x) & \text{if } x < 0 \\
\psi_+(x) & \text{if } x > 0 \end{cases}$

where $\psi_-, \psi_+ \in C^\infty(\mathbb{R}^1)$ are solutions of

$\psi''(x) + k\psi(x) = 0$, $x \in \mathbb{R}^1$,

satisfying certain initial conditions

$\psi_-(x_0) = y_0$, $\psi'_-(x_0) = y_1$
\[\psi_+(x_1) = z_0, \quad \psi'_+(x_1) = z_1,\]

where \(-\infty \leq x_0 \leq 0 \leq x_1 \leq \infty\) and \(y_0, y_1, z_0, z_1 \in C^1\) are given and the vectors

\[
\begin{pmatrix}
y_0 \\
y_1 \\
z_0 \\
z_1
\end{pmatrix}, \quad \begin{pmatrix}
y_0 \\
y_1
\end{pmatrix},
\]

might be in a certain relation.

As known, [44], that is the situation in the case of \(m = 1\) and \(x_0 = x_1 = 0\), when the junction relation in \(x = 0\) between \(\psi_-\) and \(\psi_+\) is given by

\[
\begin{pmatrix}
\psi_+(0) \\
\psi'_+(0)
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \psi_-(0) \\
\psi'_-(0)
\end{pmatrix}
\]

In the case of an arbitrary positive power \(m < (0, \infty)\), the following three problems arise:

1) to define the power \((\delta(x))^m, x \in R^1\), of the Dirac \(\delta\) distribution,
2) to prove that the hypothesis (3) is correct, and
3) to obtain a junction relation generalizing (4).

The first problem is solved in §5, where a special case of the Dirac algebras constructed in chapter 2 will be employed. The solution of the second problem results from theorem 4 in §5, and is based on the smooth representation of \(\delta\) constructed in §4. The third problem will be the one solved first, using a standard 'weak solution' approach presented in §3. That approach will also suggest the way the first two problems can be solved.

The junction relations in \(x = 0\) between \(\psi_-\) and \(\psi_+\), will be:

\[
\begin{pmatrix}
\psi_+(0) \\
\psi'_+(0)
\end{pmatrix} = Z(m, \alpha) \begin{pmatrix} \psi_-(0) \\
\psi'_-(0)
\end{pmatrix}
\]

where

\[
\begin{align*}
Z(m, \alpha) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{for} \quad m \in (0,1), \quad \alpha \in R^1, \\
Z(1, \alpha) &= \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad \text{for} \quad \alpha \in R^1 \quad \text{(see [44] and (4))} \\
Z(2, - (\eta^2)^2) &= \begin{pmatrix} (-1)^{\sigma} & 0 \\ 0 & (-1)^{\eta} \end{pmatrix}, \quad \text{for} \quad \eta = 0, 1, 2, \ldots \\
Z(m, \alpha) &= \begin{pmatrix} \sigma & 0 \\ K & \sigma \end{pmatrix}, \quad \text{for} \quad m \in (2, \infty), \quad \alpha \in (-\infty, 0)
\end{align*}
\]

with \(\sigma = \pm 1\) and \(-\infty \leq K \leq +\infty\) arbitrary.