1. Locally Convex Structures

1.1. In what follows, the letter $\mathbb{K}$ will always denote either the field $\mathbb{R}$ of the real numbers or the field $\mathbb{C}$ of the complex numbers.

A subset $A$ of a vector space $E$ over $\mathbb{K}$ is said to be: absorbing if for all $x \in E$ there exists $\alpha > 0$ such that $x \in \lambda A$ whenever $\lambda \in \mathbb{K}$ satisfies $|\lambda| < \alpha$; balanced if $x \in A$ and $|\lambda| \leq 1$ imply $\lambda x \in A$; convex if $x, y \in A$ and $\alpha + \beta = 1$, $0 \leq \alpha, \beta \leq 1$ imply $\alpha x + \beta y \in A$. A locally convex structure on $E$ is defined by a non-empty collection $\mathcal{V}$ of non-empty subsets of $E$ which satisfy:

1. if $V \in \mathcal{V}$ and $W \supset V$, then $W \in \mathcal{V}$,
2. every $V \in \mathcal{V}$ is absorbing,
3. any finite intersection of sets belonging to $\mathcal{V}$ contains a balanced, convex set belonging to $\mathcal{V}$,
4. if $V \in \mathcal{V}$ and $\lambda > 0$, then $\lambda V \in \mathcal{V}$.

A vector space equipped with a locally convex structure is called a locally convex space. Any $V \in \mathcal{V}$ contains the origin $0$ (by 3); any finite intersection of sets belonging to $\mathcal{V}$ belongs to $\mathcal{V}$ (by 3 and 1); if $V \in \mathcal{V}$ and $\lambda \neq 0$, then $\lambda V \in \mathcal{V}$ (by 3, 4 and 1).

1.2. Let $E$ and $F$ be two locally convex spaces whose structures are defined by the collections $\mathcal{V}$ and $\mathcal{W}$, respectively. A linear map $f: E \to F$ is a morphism of locally convex spaces if $f^{-1}(W) \in \mathcal{V}$ for all $W \in \mathcal{W}$. The morphisms $f: E \to F$ form a vector space $\mathcal{L}(E,F)$. The identity map $1_E$ belongs to $\mathcal{L}(E,E)$; if $E,F,G$ are three locally convex spaces, and $f \in \mathcal{L}(E,F)$, $g \in \mathcal{L}(F,G)$, then $g \circ f \in \mathcal{L}(E,G)$: the locally convex spaces form a category.

1.3. Remark. Let $\tau$ be a topology on $E$ such that the maps $(x,y) \mapsto x + y$ from $E \times E$ into $E$ and $(\lambda, x) \mapsto \lambda x$ from $\mathbb{K} \times E$ into $E$ are continuous and assume that $0$ has a fundamental system of neighborhoods which consists of convex sets. Then the collection $\mathcal{V}$ of all neighborhoods of $0$ satisfies conditions (1) - (4). Conversely, if a locally convex
structure is defined by a collection \( \mathcal{V} \), then the sets \( x + \mathcal{V} \), where \( \mathcal{V} \in \mathcal{V} \), will be the collection of all neighborhoods of the point \( x \) for a topology for which the maps \( (x, y) \mapsto x + y, (\lambda, x) \mapsto \lambda x \) are continuous and each point possesses a fundamental system of convex neighborhoods. The morphisms are then the continuous linear maps. This method of introducing locally convex spaces (i.e., locally convex topological vector spaces) avoids introducing preliminary topological concepts [75].

1.4. Let \( \mathcal{V} \) be a collection of absorbing, balanced, convex subsets of the vector space \( E \). Then the collection \( \mathcal{V} \) of all sets \( \lambda \mathcal{V} \), where \( \lambda > 0 \) and \( \mathcal{V} \) contains a finite intersection of sets belonging to \( \mathcal{V} \), or equivalently the collection of all sets \( \mathcal{V} \) which contain a finite intersection of sets of the form \( \lambda \mathcal{W} \), \( \lambda > 0 \), \( \mathcal{W} \in \mathcal{V} \), satisfies (1) - (4) and so defines a locally convex structure on \( E \), said to be generated by \( \mathcal{V} \). Two different collections \( \mathcal{V} \) can generate the same locally convex structure. The collection of all balanced, convex subsets belonging to \( \mathcal{V} \) generate the locally convex structure defined by \( \mathcal{V} \).

1.5. A semi-norm on \( E \) is a map \( p \) from \( E \) into the set \( \mathbb{R}_+ \) of positive real numbers which satisfies \( p(x + y) \leq p(x) + p(y) \) and \( p(\lambda x) = |\lambda| p(x) \) for all \( x, y \in E \), \( \lambda \in \mathbb{K} \). The closed semi-ball \( \{ x \mid p(x) \leq 1 \} \) and the open semi-ball \( \{ x \mid p(x) < 1 \} \) are then absorbing, balanced, convex sets. Thus, given a collection \( \mathcal{P} \) of semi-norms on \( E \), the closed semi-balls (or equivalently the open semi-balls) pertaining to the \( p \in \mathcal{P} \) generate a locally convex structure on \( E \). Conversely, every locally convex structure can be so generated since the gauge ("Minkowski functional") \( p_V(x) = \inf \{ \lambda \mid \lambda > 0, x \in \lambda \mathcal{V} \} \) of an absorbing, balanced, convex set \( V \) is a semi-norm.

1.6. Example. Let \( X \) be a completely regular (Hausdorff) topological space. A Nachbin family \( \mathcal{U} \) on \( X \) is a collection of positive, upper semi-continuous functions on \( X \) such that for \( v_1, v_2 \in \mathcal{U} \) and \( \lambda > 0 \) there exists \( v \in \mathcal{U} \) with \( \max(\lambda v_1(x), \lambda v_2(x)) \leq v(x) \), \( x \in X \). Denote by \( \mathcal{C} \mathcal{U}(X) \) the vector space of all continuous functions \( f \) on \( X \) such that \( v f \) is bounded for all \( v \in \mathcal{U} \). The family of semi-norms \( (p_v)_{v \in \mathcal{U}} \) given by \( p_v(f) = \sup_{x \in X} |v(x)f(x)| \) defines a locally convex structure on \( \mathcal{C} \mathcal{U}(X) \).