In this note we construct a measurable set \( D \subset [0,\infty) \times \Omega \), a 3-dimensional Bessel process, \( X \), and a filtration, \( \{ F^B_t \} \), containing the canonical filtration, \( \{ F^X_t \} \), of \( X \) satisfying the following properties:

(i) \( X \) is an \( \{ F^B_t \} \)-semimartingale.
(ii) \( D \) is an \( \{ F^X_t \} \)-progressively measurable set, i.e.,
    \[ D \cap ([0,t]) \in \text{Borel} \left( [0,t] \times F^X_t \right) \text{ for all } t \geq 0. \]
(iii) \( \int_0^t I_D \, dX = X(t) \), where the left side is interpreted with respect to \( \{ F^X_t \} \), and \( I_D \) denotes the indicator function of \( D \).
(iv) \( \int_0^t I_D \, dX \) is an \( \{ F^B_t \} \)-Brownian motion when the stochastic integral is taken with respect to \( \{ F^B_t \} \).

As the local martingale part of \( X \) with respect to either filtration will be a Brownian motion (since \( [X](t) = t \)) , \( \int_0^t I_D \, dX \) may be defined in the obvious way even though \( D \) will not be predictable.

Let \( B \) be a \( 1 \)-dimensional Brownian motion on a complete \((\Omega, F, P)\). If \( M(t) = \sup_{s \leq t} B(s) \), \( Y = M - B \) and \( X = 2M - B \), then \( Y \) is a reflecting Brownian motion, and \( X \) is a 3-dimensional Bessel process by a theorem of Pitman [4]. \( \{ F^X_t \} \), respectively \( \{ F^B_t \} \), will denote the smallest filtration, satisfying the usual conditions, that makes \( X \), respectively \( B \), adapted. \( F^X_t \leq F^B_t \) is clear, and since \( M(t) = \inf_{s \geq t} X(s) \), the inf being assumed at the next zero of \( Y \), we must have \( F^X_t \leq F^B_t \) for \( t > 0 \), as \( M(t) \) cannot be \( F^X_t \)-measurable. Finally, define

\[ D = \{ (t, \omega) \mid \lim_{n \to \infty} \lim_{k \to \infty} I(X(t+2^{-k}) - X(t+2^{-k-1}) > 0) = 1/2 \} \].
Property (i) is immediate and for (ii), fix $t \geq 0$ and note that

$$D \cap [0,t] = \{(t) \times D(t)\} \cup \{(s,\omega)\mid s \leq t - 2^{-N}\},$$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=N}^{\infty} I(X(s+2^{-k}) - X(s+2^{-k-1}) > 0) = 1/2 \in \text{Borel}([0,t]) \times F_t^X.$$

Here $D(t)$ is the $t$-section of $D$. To show (iii) choose $t > 0$ and note that

$$X(t+2^{-k}) - X(t+2^{-k-1}) = B(t+2^{-k-1}) - B(t+2^{-k})$$

for large $k$ a.s.

Therefore the law of large numbers implies that

$$\text{(1)} \quad P((t,\omega) \in D) = 1 \quad \text{for all} \quad t > 0.$$

The canonical decomposition of $X$ with respect to $\{F_t^X\}$ is (see McKean [3])

$$\text{(2)} \quad X(t) = W(t) + \int_0^t X(s)^{-1} ds,$$

where $W$ is an $\{F_t^X\}$-Brownian motion. Therefore with respect to $\{F_t^X\}$ we have

$$\int_0^t I_D dX = \int_0^t I_D dW + \int_0^t I_D X_s^{-1} ds = X(t) \quad \text{a.s. (by (1))}.$$

It remains only to prove (iv). If

$$T(t) = \inf\{s\mid M(s) > t\},$$

we claim that

$$\text{(3)} \quad P((T(t),\omega) \in D) = 0 \quad \text{for all} \quad t \geq 0.$$

Choose $t \geq 0$ and assume $P((T(t),\omega) \in D) > 0$. Since $X(T(t)+\cdot) - X(T(t))$ is equal in law to $X(\cdot)$, the 0-1 law implies that

$$P((T(t),\omega) \in D) = 1.$$

The dominated convergence theorem and Brownian scaling imply