VECTOR LATTICES OF UNIFORMLY CONTINUOUS FUNCTIONS AND SOME
CATEGORICAL METHODS IN UNIFORM SPACES

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This somewhat expository paper is organized around the diagram

\[
\text{unif} \xrightarrow{C} \text{C-spaces} \xleftarrow{G} \mathcal{E}\text{-spaces} \xrightarrow{F} \mathcal{L}
\]

with categories: \text{unif}, of separated uniform spaces; \mathcal{L}, of "semi-simple vector lattices with unit"; \mathcal{E}\text{-spaces}, whose objects are pairs \((L,X)\), where \(L\) is a point-separating vector lattice of real-valued functions on the set \(X\), containing the constant function 1 (so \(L \subseteq \mathcal{L}\)), and a morphism \((L_1,X_1) \xrightarrow{\varphi} (L_2,X_2)\) is an algebraic homomorphism \(L_1 \xrightarrow{\varphi} L_2\) for which there is a (necessarily unique) map \(X_1 \xleftarrow{\varphi'} X_2\) such that \(\varphi(f) = f \varphi'\) for \(f \in L_1\); a \(\mathcal{E}\)-space is an \(\mathcal{E}\)-space of the form \((C(\mu X),X)\), where \(\mu X \in \text{unif}\) and \(C(\mu X)\) is all uniformly continuous \(\mu X \rightarrow \mathbb{R}\) (\(\mathbb{R}\) being the usual real line).

and functors: \(C\), just described; \(G(L,X) = (C(\mu L,X),X)\), where \(\mu L\) is the weak uniformity generated by \(L\); \(F\), the obvious forgetful functor; \(H\), which represents \(L \subseteq \mathcal{L}\) as an \(\mathcal{E}\)-space \((L',\mathcal{U}(L))\), where \(\mathcal{U}(L)\) is a certain space of ideals (or Homomorphisms) of \(L\).

The dotted arrow stands some methods of generating subcategories of \text{unif} from subcategories of \text{C-spaces} (or of \(\mathcal{L}\)). (There is a functor in the position of the dotted arrow, "quasi-adjoint" to \(C\).)

Note that if \((L_1,X_1) \xrightarrow{\varphi} (L_2,X_2)\) is an \(\mathcal{E}\)-space morphism, with inducing map \(X_1 \xleftarrow{\varphi'} X_2\), then \(\mu_{L_1} X_1 \xleftarrow{\varphi'} \mu_{L_2} X_2\) is uniformly
continuous. The converse holds for $C$-spaces: each uniformly continuous $\mu X \xrightarrow{\varphi'} \nu Y$ induces a morphism $(C(\mu X), X) \xrightarrow{\varphi} (C(\nu Y), Y)$. This makes the category of $C$-spaces the "opposite" of the full subcategory of $\text{uf}$ of all $\mu X$ where $\mu$ is weak generated by some $pR$-valued functions. This latter category is $O(pR)$, all subspaces of uniform powers of $pR$, and we have a reflecting functor $\text{uf} \xrightarrow{C} O(pR)$.

The paper contains at least ten reflections and twenty-two coreflections, so we state the definition: Subcategory $\mathfrak{Q}$ of $G$ is reflective in $G$ if to $A \in G$ corresponds $A \xrightarrow{b_A} bA \in \mathfrak{Q}$ such that if $A \xrightarrow{m} B \in \mathfrak{Q}$, then there is unique $bA \xrightarrow{b m} B$ with $m = (b m) \cdot b_A$. Then: $bA$ is the reflection of $A$, $b_A$ is the reflection map, and $b$ is a functor, the reflector. (We shall assume each $b_A$ epic). Coreflection is defined dually. (For these ideas in topology, see Kennison, Herrlich, Herrlich-Strecker (1,2).)

Examples. $\text{uf} \xrightarrow{C} O(pR)$, $\text{uf} \xrightarrow{e} \text{Sep}$, $\text{uf} \xrightarrow{Y} \Gamma = \text{complete spaces}$ are reflectors: $\text{Sep}$ consists of "separable" uniform spaces (with basis of countable covers) and $e \mu X$ has basis of countable $\mu$-covers; $\gamma$ is completion. $G$ is a reflector in $L$-spaces; rings with identity are reflective in $L$. $\text{uf} \xrightarrow{\alpha T}$ Fine spaces is a coreflector, where $T \mu X$ denotes the underlying topological space and $\alpha$ is the fine uniformity.

We can outline the paper. Section 1 gives a characterization of $C$-spaces among $L$-spaces of Fenstad (1,2). Section 2 shows that $(F, H)$ is an equivalence between complete $L$-spaces and $L$, thus as equivalence between complete $C$-spaces and a certain $L_C$ (described by Section 1), thus a duality (or contravariant equivalence) between $O(pR) \cap \Gamma$ is $R(pR)$, the reflective hull of $pR$, consisting of all closed subspaces of powers of $pR$.) This duality follows Isbell (1) and Fenstad (2), but the explicit statement of it seems new. Section 3 considers special properties of $C(\mu X)$'s (e.g., ring) which "are"