THE GEOMETRICAL PROPERTIES OF THE REDUCED DOUBLE CUSP.

by

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ABSTRACT: The Bifurcation Sets of the Reduced Double Cusp

\[ V = A(x^4 + y^4 - 6x^2y^2) + B(x^3 - 3xy^2) + C(x^2 + y^2) - ux - vy \]

were drawn to investigate the possibility of the existence of three-, four- and five-fold symmetry in the projected envelope. While three- and four-fold symmetry could be demonstrated, it was impossible to demonstrate five-fold symmetry. However, modifying one of the parametric equations in a heuristic way did permit the production of a figure with five-fold symmetry.

This work was begun when both authors were at the Institute of Mathematics, University of Warwick, Coventry CV4 7AL, Warwickshire, England.
INTRODUCTION

René Thom (1), in discussing the properties of the so-called "Double Cusp" states that the unfolding of the Double Cusp Singularity has certain symmetry properties, which he enumerates. It is one purpose of this paper to investigate the validity of those statements using the graphical techniques that were developed earlier (Woodcock and Poston, 2, 3, 4). The version of the Double Cusp described by Thom is given by the following equation:

\[ v = A(x^4 + y^4 - 6x^2y^2) + B(x^3 - 3x^1y^2) + C(x^2 + y^2) - ux - vy \]  

(1)

(we will term the singularity given by equation (1), the Reduced Double Cusp) We will investigate the three regions of interest that Thom has considered, namely:

1. When A is zero and B and C are non-zero, Thom says that "the curve of critical values in the (u,v) plane is a hypocycloid with three cusps, \( H_3 \), and hence that it has tertiary symmetry". In this case, the equation reduces to that of the Elliptic Umbilic, which, indeed does have tertiary symmetry (Woodcock and Poston, (3)).

2. When B is zero and A and C are non-zero, "the projection of the critical curve in the u,v plane is an hypocycloid with four cusps, or quaternary symmetry".

3. When C is zero and A and B are non-zero, "the corresponding curve in the (u,v) plane is a cycloidal curve which defines the stellated pentagon (the pentagram): it is invariant under rotation of angle \( 2\pi/5 \)."

We will draw the appropriate sections of the Reduced Double Cusp Catastrophe control surface in order to study the validity of these statements. Furthermore, we will show sections other than those described above to demonstrate how one type of critical curve may be transformed into another.