The Strong Markov Property for Canonical Wiener Processes

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1. Canonical Wiener Processes

By a canonical Wiener process of variance \( \sigma^2 \geq 1 \) indexed by the positive half-line \( \mathbb{R}^+ \), we mean a triple \((P, Q, \psi)\), where \( P, Q \) are functions from \( \mathbb{R}^+ \) to the self-adjoint operators in a Hilbert space \( H \) and \( \psi \) is a unit vector in \( H \), with the following properties.

\( \mathcal{O} \). \([P(s), P(t)] = [Q(s), Q(t)] = 0, [P(s), Q(t)] = -i \min(s, t)\).

1. \( P(0) = Q(0) = 0 \).

2. For disjoint intervals \( \Delta_i = (a_i, b_i] \), \( \Delta_i = (a_n, b_n] \subset \mathbb{R}^+ \), denoting by \((p_i, q_i)\) the canonical pair \((\sqrt{b-a})(P(b) - P(a)), \sqrt{b-a}(Q(b) - Q(a))\)

when \( \Delta = (a, b] \), the canonical pairs \((p_\Delta, q_\Delta)\), \( \ldots, (p_{\Delta_n}, q_{\Delta_n})\) are independent and identically normally distributed with zero means and covariance matrix \( \frac{1}{2} \sigma^2 I_2 \) in the state determined by \( \psi \).

Here the commutation relation \([X, Y] = -ic\), where \( c \) is a real number and \( X, Y \) are unbounded self-adjoint operators means the corresponding Weyl relation \( e^{ixX} e^{iyY} = e^{-icy} e^{ixX} \). A canonical pair is a pair of self-adjoint operators \( (p, q) \) satisfying \([p, q] = -iI\) and the definitions of independence, identity of distribution, normal distribution, mean and covariance matrix for canonical pairs are as in [3], [2].

Canonical Wiener processes indexed by the unit interval were introduced in [2] and shown to have some properties analogous to those of classical Wiener processes in [1], [2].

A canonical Wiener process \((P_0, Q_0, \psi_0)\) with \( \sigma^2 = 1 \) is obtained from the Fock representation of the canonical commutation relations over the real Hilbert space \( \mathcal{F} \) of square-integrable functions on \( \mathbb{R}^+ \) by taking \( \psi_0 \) to be the Fock vacuum vector and setting \( P_0(t) = \pi(\chi_{[0, t]}), Q_0(t) = \pi(\chi_{[0, t]}), \) where for \( f \in \mathcal{F}, \pi(f), \overline{\pi(f)} \) are the canonical field operators and \( \chi_{[0, t]} \) is the indicator function of \([0, t]\). When \( \sigma^2 > 1 \), a canonical Wiener process \((P, Q, \psi)\) can be constructed in the Hilbert space tensor product \( H = H_0 \otimes H_0 \) of two copies of Fock space by taking \( \psi \) to be \( \psi_0 \otimes \psi_0 \) and setting

\[ P(t) = 2^{-\frac{1}{2}}(\alpha P_0(t) \sigma 1 + \alpha^{-1} Q_0(t)), Q(t) = 2^{-\frac{1}{2}}(\alpha^{-1} Q_0(t) \sigma 1 - \alpha 1 P_0(t)) \]

(1.1)

where \( \alpha \) is a real number such that \( \alpha^2 + \alpha^{-2} = 2\sigma^2 \). These processes are cyclic, meaning that repeated action on the state vector by the constituent operators of the process yields a total set of vectors, and every cyclic process is unitarily equivalent to one of this type.

We denote by \( N \) the von Neumann algebra generated by the spectral projections of the constituent operators of the process \((P, Q, \psi)\), and
for \( \lambda \geq 0 \) by \( N_{\lambda} \) (resp. \( N_{-\lambda} \)) the pre- (resp. post-) \( \lambda \) algebra, generated by the spectral projections of \( P(t),Q(t), t \leq \lambda \) (resp. of \( P(t+\lambda)-P(\lambda), Q(t+\lambda)-Q(\lambda), t \geq 0 \)). \( N_{\lambda}, N_{-\lambda} \) are independent in the sense that if \( A \in N_{\lambda}, B \in N_{-\lambda} \) then \( A, B \) commute and \( \langle AB^\dagger, \gamma \rangle = \langle A^\dagger, \gamma \rangle \langle B, \gamma \rangle \).

A positive self-adjoint operator \( T \) with spectral resolution \( T = \int_0^\infty \lambda dE(\lambda) \) is a Markov time if \( E(\lambda) \in N_{\lambda} \) for all \( \lambda \geq 0 \).

2. Existence of \( P_T, Q_T \)

Analogy with the strong Markov property for a classical Wiener process [5] suggests that if \( T = \int_0^\infty \lambda dE(\lambda) \) is a Markov time for the cyclic canonical Wiener process \( (P, Q, \gamma) \) then \( P_T, Q_T, \gamma \) is also a canonical Wiener process, where formally

\[
P_T(t) = \int_0^\infty (P(t+\lambda)-P(\lambda)) dE(\lambda), \quad Q_T(t) = \int_0^\infty (Q(t+\lambda)-Q(\lambda)) dE(\lambda).
\]

To give meaning to (2.1) we first write down the equivalent forms

\[
e^{ixP_T(t)} = \int e^{i\lambda} dE(\lambda), \quad e^{ixQ_T(t)} = \int e^{i\lambda} dE(\lambda).
\]

and observe that the integrands in (2.2), (2.3) belong to \( \lambda N \) whereas the integrator belongs to \( \lambda N \), suggesting that the integrals be defined as strong operator limits of 'backward' Riemann-Stieltjes sums

\[
K(\lambda) = \frac{1}{n} \sum_{j=1}^n e^{i\lambda} e^{-i\lambda} (E(\lambda_j)-E(\lambda_{j-1})),
\]

\[
L(\lambda) = \frac{1}{n} \sum_{j=1}^n e^{i\lambda} e^{-i\lambda} (E(\lambda_j)-E(\lambda_{j-1})).
\]

where \( b > 0 \) and \( \lambda = (0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n = b) \) is a partition of \([0, b] \).

Theorem 2.1 As \( \max(\lambda_j - \lambda_{j-1}) \to 0 \) and \( b \to \infty \) \( K(\lambda), L(\lambda) \) converge strongly to operators \( U, V \), respectively. Moreover for fixed \( t, x \mapsto U_{x,t}, \quad x \mapsto V_{x,t} \), are strongly continuous one-parameter unitary groups whose infinitesimal generators \( P_T(t), Q_T(t) \) satisfy the defining properties (0), (1) of a canonical Wiener process.

Lemma 2.2 For fixed \( t, (x, \lambda) \mapsto e^{ixP(t+\lambda)} e^{-i\lambda} \) is strongly continuous on \( RxR_+ \).

Proof By (1.7) in the case \( \sigma^2 > 1 \) these operator-valued functions are tensor products of corresponding functions for the Fock case \( \sigma^2 = 1 \). Hence it is sufficient to consider the latter case. Using the Fock vacuum expectation functional \( \langle e^{i\pi(f)} \gamma, \gamma \rangle = e^{-\frac{1}{2} \| f \|_2} \), we have after some manipulations, for arbitrary \( f, g \in \mathfrak{g} \),

\[
\| (e^{i\pi(f)} e^{-i\lambda} - e^{i\pi(f)} e^{-i\lambda})_g \|_2 = 2(1 - \cos(\lambda t + \mu) - \eta I_{(\mu, t+\mu)} g) e^{-\frac{1}{2} \| \gamma \|_2}.
\]

From this it is clear that \( (x, \lambda) \mapsto e^{ixP(t+\lambda)} e^{-i\lambda} \) is continuous. Since \( \gamma \) is cyclic vectors of the form \( e^{i\pi(f)} \), \( \gamma \) are total in \( H_0 \) and the result follows.

Lemma 2.3 There exists a total set \( S \) of which \( \gamma \) is an element such