VECTOR MEASURES AND THE ITO INTEGRAL

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It seems appropriate at a conference on vector-valued measures to look at one of the best examples of one: the stochastic integral. It is not obvious from the usual construction that the stochastic integral is at all connected with vector measures. It looks more like a form of Riemann integral, but as such it has some puzzling properties. For instance, in the approximating Riemann sums, the integrand must be evaluated at the left-hand endpoints of the intervals. Why not in the middle? Why not at the right? These are natural questions, but one feels that they are of the kind that would not come up if it were really a Riemann integral in any reasonable sense. Once we see that there is in fact a vector-valued measure and that the Ito integral is really an integral in the usual sense with respect to this measure, they will practically disappear.

The approach we will describe below is, except for some minor modifications, due to J. Pellaumail. Since the detailed proofs are carefully set out in [2] and [4], we will be more concerned with showing informally why things ought to be true than with proving that they actually are true. We will also confine ourselves to the most elementary part of the theory; the reader will find further - and deeper - developments in [2] and [4].

1. ITO MEASURE.

Let \{B_t, t \geq 0\} be a standard Brownian motion. We want to define

(1) \[ \int_0^t \psi_s \, dB_s \]

for suitable integrands \(\psi\). The first difficulty is that \(t \to B_t\) is not of bounded variation - it is even nowhere differentiable - and there is no way to write it as the difference of two increasing functions. Thus we can't define (1) as a Stieltjes integral. In fact, consider the famous integral

(2) \[ \int_0^t B_s \, dB_s = B_t^2 - \frac{1}{2} t. \]

If this were a Stieltjes integral, we would not get the \(\frac{1}{2} t\) term. But let's look at it from the point of view of a vector integral.

If we want to write (1) and (2) as integrals with respect to a vector measure \(\mathcal{B}\), we first have to find the measure. To do this we need to answer the following two questions.

(i) On what \(\sigma\)-field is \(\mathcal{B}\) defined?

(ii) In which space does it take its values?

To answer (i), let's consider which sets we can measure. These certainly
include the half-open intervals of the form \((s, t]\) - think of \(B_t\) as the distribution function of the measure \(B\) - for the \(B\)-measure of \((s, t]\) should be \(B_t - B_s\). But this is not enough. We will need to integrate random functions, as in (2) for instance, so we will need to be able to measure random intervals. Let \(S \leq T\) be bounded stopping times and let

\[
(S, T] = \{(t, w) : S(w) < t \leq T(w)\}
\]

be a stochastic interval. Then we can put \(B(S, T] = B_T - B_S\). Notice that a stochastic interval is a subset of \(\mathbb{R}_+ \times \Omega\), not of \(\mathbb{R}_+\), so our \(\sigma\)-field will be composed of subsets of \(\mathbb{R}_+ \times \Omega\), not of the line. The simplest \(\sigma\)-field to take is surely the \(\sigma\)-field \(\mathcal{P}\) generated by the stochastic intervals of the form \((S, T]\), where \(S\) and \(T\) are bounded stopping times, which is called the \(\sigma\)-field of predictable sets. It has the property that any process \(X = \{X_t, t \geq 0\}\) which is adapted to \((\mathcal{F}_t)\) and left continuous as a function of \(t\) is \(\mathcal{P}\)-measurable. We say that a \(\mathcal{P}\)-measurable process is predictable. Thus \(B_t\) itself is predictable so that, for instance, \(B\) is large enough to allow us to define the integral (2).

(We should underline the fact that a process \(X\) is really a function of two variables, \(X_t(w)\), even though the \(w\) is usually suppressed, so that \(X\) is \(\mathcal{P}\)-measurable as a function of the two variables).

Now that we have decided that \(B\) will be a measure on \(\mathcal{P}\), let's see what vector space is involved. It turns out that there are many possibilities, and the choice depends somewhat on the use one wants to make of the integral, and somewhat on his taste. Pellaumail and Métrivier use \(L^p(\Omega, \mathcal{F}, \mathbb{P})\) and \(L^0(\Omega, \mathcal{F}, \mathbb{P})\), this last being the space of all random variables, given the topology of convergence in measure. These are probably the best spaces to take in most cases, but in order to understand the essence of the stochastic integral, we would like to take a vector space which expresses its most important properties. We have two properties in mind. The first is that \(t \mapsto \int_0^t \phi_s \ d B_s\) should be continuous. Indeed, Itô first introduced the stochastic integral in order to solve stochastic differential equations. The solutions of these are diffusions, which model physical particles which diffuse through a liquid and whose motions are therefore continuous. It would have been embarrassing had the stochastic integrals turned out to be discontinuous!

The second property is that the process \(\{\int_0^t \phi_s \ d B_s, t \geq 0\}\) is a martingale. One has to work with stochastic integrals to appreciate how vital this is. It is one of the main reasons that they can be defined for such a large class of integrands, and it comes up continually in applications.

Let's consider the square-integrable case, which is always the easiest to handle in this theory. Let \(\mathcal{M}_2^2\) be the set of all right-continuous martingales