DEFECT CORRECTIONS AND MULTIGRID ITERATIONS

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Introduction

In several of his publications since 1978, Brandt describes a modification of the general multigrid algorithm which he calls "truncation extrapolation" or "\(\tau\)-extrapolation" (see, e.g., [1]). In their analysis of this idea, the authors of this paper reinterpreted Brandt's \(\tau\)-extrapolation as a defect correction step ([2]). Independently, W. Hackbusch had suggested a combination of the multigrid and defect correction approaches in [3], see also [4].

In the following, we will at first demonstrate how any defect correction iteration may be combined with an iterative procedure for its effective solution operator and derive a bound on the convergence factor for the combined iterative process. We will then consider the situation when the iterative solution procedure consists of multigrid cycles. Various implementation versions of a defect correction multigrid cycle will be discussed, particularly for the case where defect correction is to achieve a higher order approximation to the differential equation. We will also regard some algorithmic and quantitative details in the context of model problems. Finally we will present some numerical experiences.

Our approach permits the use of standard multigrid software in situations more general than those for which the software was designed. This will extend the applicability of existing and coming multigrid packages and thus reduce programming costs for many applications.

1. Defect Correction with an Iterative Solution Procedure

The principle of defect correction may be summarized thus (cf., e.g., [5]): We wish to solve the problem

\[
P u = c \quad (1.1)
\]
but we only have an efficient solution procedure for the related problem

\[ \tilde{F} v = \tilde{c} ; \]  

(1.2)

however, we are able to evaluate the direct mapping \( F \). Then the following two iterative processes are natural (see [5]):

(A) \[ u^{(i+1)} := u^{(i)} - (\tilde{F}^{-1} F u^{(i)} - \tilde{F}^{-1} c) \]

or

(B) \[ \tilde{F} u^{(i+1)} := \tilde{F} u^{(i)} - (F u^{(i)} - c) . \]

If \( \tilde{F} \) is a linear mapping (\( F \) may still be nonlinear), then (A) and (B) collapse into the familiar

\[ u^{(i+1)} := u^{(i)} - \tilde{F}^{-1} (F u^{(i)} - c) . \]  

(1.4)

Convergence of these processes depends on the contractivity of the mappings

(A) \( I - \tilde{F}^{-1} F \) or \( (B) \ I - F \tilde{F}^{-1} ; \)

in the case of linear \( F \) these two mappings are related by the similarity transformation

\[ I - \tilde{F}^{-1} F = \tilde{F}^{-1} (I - F \tilde{F}^{-1}) \tilde{F} . \]

(Here and in the following, we always assume that solutions of equations exist and are unique at least locally so that the notion of an inverse mapping makes sense.)

We will now assume that the efficient solution procedure for problems (1.2) is itself an iterative process and of the defect correction type; let us denote it by

\[ v^{j+1} := v_j - K (\tilde{F} v_j - \tilde{c}) . \]  

(1.5)

In conjunction with the iterative process (1.4) or (1.3), one will not want to use more than one or a few iterations (1.5) - with an appropriate initial value \( v_0 \) - in place of \( \tilde{F}^{-1} \), even if (1.5) does not converge very rapidly towards the solution \( \tilde{F}^{-1} \tilde{c} \) of (1.2).