The Convergence Rate of a Multigrid Method
with Gauss-Seidel Relaxation for the Poisson Equation

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The numerical solution of the Poisson equation is treated by a multi-
grid method for a uniform grid. The convergence rate can be estimated
even for the iteration with a V-cycle independently of the shape of
the domain as long as it is convex and polygonal. The smoothing effect
of the Gauss-Seidel relaxation is described by a discrete seminorm
which is weaker than the energy norm.
1. Introduction

The linear equations which result from the discretization of elliptic boundary value problems typically have matrices with large condition numbers. This is due to the fact that the spectra of the associated differential operators are not bounded. Therefore the efficient numerical solution of the equations requires facilities for the adequate treatment of the smooth portions of the solution as well as for the non-smooth ones. Possibly one needs algorithms which combine two different devices for this aim. It has turned out during the last years that multigrid methods have the desired properties and can solve the equations very effectively [5,6,10].

For the same reason the numerical analysis of multigrid methods can probably not be done effectively if one restricts oneself to the use of one norm. (At a first glance [3] seems to be a counter-example.) One has to measure at least the amount and the smoothness of an approximate solution or of its error, respectively. This has been done in two different ways, and two different types of convergence proofs for multigrid methods are found in the literature.

A. Fourier Method: Exact bounds for the convergence rates are established by Fourier analysis. The results show the good performance of the method [5,8,11]. But the analysis is restricted to rectangular domains.

B. Asymptotical Analysis: A convergence rate bounded away from 1 is established for sufficiently many smoothing steps. The domains may be arbitrary as long as they are consistent with the regularity of the boundary value problem [1,6,9]. Usually norms of Sobolev type are used to control the process.

There is still the unanswered question whether and how the (good) performance depends on regularity and on the shape of the boundary. This question is important not only with the use of multigrid codes but also with respect to the reliability of the local mode analysis [5]. Though this heuristic method neglects the influence of boundaries and boundary conditions it often yields results which are close to the observed numbers.

Here we will consider a multigrid method for the discretization of the Poisson equation with Dirichlet boundary conditions for which the questions above can be partially answered (though we are still far from an answer in the general case).