§1. Introduction

In the theory of transformation groups the smooth theory is certainly the most tractable. However, not all operations that one would like to perform remain in the smooth category. For example, analysis of the orbit space usually must go outside the smooth category. Furthermore, one would also like to study symmetries of interesting geometric spaces which often fail to be locally Euclidean such as spaces having manifolds as ramified coverings and analytic spaces.

It has long been recognized that cohomology manifolds or generalized manifolds encompass all manifolds as well as many typical analytic and ramified spaces. The most important feature of generalized manifolds, from the point of view of transformation groups, is that one is often able to work with orbit spaces (cf. [8]). In addition, one does have characteristic classes and with care one can often define workable invariant tubular neighborhoods of fixed point sets.

In this paper we develop and exploit some of these ideas to prove results which even when specialized to the smooth category seem to be unobtainable from standard smooth methods. We introduce the notion of nicely embedded fixed points. This enables us to find invariant closed tubular neighborhoods which are cohomology fiber spaces [2] over the fixed point sets. The mapping cylinder of the orbit map is an important construction. For circle actions, without fixed points on generalized manifolds, this mapping cylinder is a generalized manifold with boundary with a natural circle action having nicely embedded fixed points identical with the orbit space. We establish by cohomological methods alone formulae for

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the Atiyah-Singer invariant $\sigma(x, (S^1, M))$, Theorems 2, 3, 4 and 5. This enables us to conclude that its value is an integer and also to give explicit geometric interpretations of it in terms of the orbit space and the mapping cylinder of the orbit map. This makes it much easier to compute than by using smooth techniques alone, e.g. \S5. A very interesting geometric interpretation of $\sigma(S^1, M^{4k-1}) = 0$ is also given in terms of fibering $M^{4k-1}$ over a circle, Theorems 4 and 5. The detailed version of these results will appear in [4].

§2. The index of manifolds with toral actions

Let us recall that a (closed) mapping cylinder neighborhood of a closed subset $A$ of a space $X$ is the closure of an open subset $U \supset A$ of $X$, a map $f$ of the frontier of $U$ onto the frontier of $A$, and a homeomorphism $h$ of the closure of $U$ minus the interior of $A$ onto the mapping cylinder of $f$ such that $h$ restricted to the frontiers of $U$ and $A$ is the identity. Note that closure $U$ minus $A$ is homeomorphic to $(\text{frontier of } U) \times [0, 1)$, with $(\text{frontier of } U) \times (0, 1)$ being an open subset of $X$. (In [6] it is shown that any two such open mapping cylinder neighborhoods (MCN) are essentially unique.)

We shall say that an $S^1$-action on an orientable closed $\mathbb{R}$-generalized $n$-manifold $M^n$ has nicely embedded fixed points if each component $F_{k, j}$ of the fixed point set $F$ has an invariant closed mapping cylinder neighborhood $T(F_{k, j})$ satisfying the conditions (i) - (iv) listed below.

Without loss of generality we may assume that $S^1$ is not acting trivially upon $M$ and $F_{k, j}$ is an $(n-2k)$-dimensional orientable generalized manifold over $\mathbb{R}$ with $k > 0$. If we let $r_t : T \to T$ denote the retraction along the "fibers," then $r_t^* 2_T = f$. We shall postulate that (i) $(r_t^* 2_T)^t(x) = S_{x}^{2k-1}$ is a $(2k-1)$-generalized manifold over $\mathbb{R}$ having the real cohomology of the $(2k-1)$-sphere, for each $x \in F_{k, j}$. This means that $r_t^{-1}(x) = S_{x}^{2k-1}$ is the cone over $S_{x}^{2k-1}$ and is consequently a generalized 2$k$-cell over $\mathbb{R}$. We shall assume that (ii) $r_t^{-1}(x)$ is invariant and the $S^1$-action on $S_{x}^{2k-1} \times (0, t)$ is independent of $t$, $0 < t < 1$. To handle how the "fibers" fit together homologically we assume that (iii) the Leray sheaf of the map $f$ is simple, coefficients in $\mathbb{R}$. Finally, to eliminate "exotic" pathology we assume that (iv) $(S^1, M^n)$ has only a finite number of distinct orbit types (always holds if $M^n$ is a $\mathbb{Z}$-generalized manifold).

Finally, if $(T^s, M^n)$ is an action of an $s$-dimensional torus on an orientable $\mathbb{R}$-generalized manifold, we shall say that it has (very) nicely embedded