\[ \xi_j = x_{j0} + \lim_{T \to \infty} \frac{1}{T} \int_0^T \phi_j(t;x_{j0}) \, dt, \]

the limit existing since \( \phi_j \) can be shown to be almost periodic using the stability assumption (2). It is easily verified that \( f \) defines a one-valued function on \( T^n \), takes orbits to straight lines, and \( f \) is continuous since the limit is uniform. The surjectivity follows from assumption (1); the argument for injectivity is less immediate, using topological group theory to show if \( G = \bigcup_{-\infty < t < \infty} x(t;x_{j0}) \) then (i) the flow structure makes \( \tilde{G} \) a topological group (by (2)), (ii) \( \tilde{G} = T^n \) as sets, (ii) \( f : \tilde{G} \to T^n \) is a group monomorphism.

With the same assumptions, it can also be proved that the flow is ergodic. Further details of the above may be found in [2].

REFERENCES:


(12) Continuous flows on the plane: techniques I A. Beck

This and the following article give an outline of the techniques used in the proofs of parts (i)', (ii)' of the main theorem of (6) (page 12 above). We consider first flows on \( E^2 \cup \{\infty\} \) with no stagnation points at all.

The principal way of studying a flow \( \phi \) in \( E^2 \) is to investigate the orbits near a periodic orbit. Let \( p(x) \) denote the period of \( x \), i.e. \( p(x) = \inf \{t \mid t > 0, \phi(x, t) = x\} \). Then \( p : X \to \mathbb{R}^+ \) is lower semi-continuous for flows on any space \( X \).
When $X = E^2$ we have:

**Lemma.** If $x_1 \rightarrow x$ and $p(x) = \infty$ then $p(x_1) = \infty$ for almost all $i$.

**Corollary.** $(x \mid 0 < p(x) < \infty)$ is closed in $E^2 \setminus F(\phi)$. (Note $F(\phi) = \{x \mid p(x) = 0\}$.)

Let $y$ be a point with $p(y) < \infty$, $y \notin F(\phi)$, and suppose

(Case 1) $y \in \omega(x)$ for some $x$, where as usual $\omega(x)$ denotes

$\{w \mid \psi(x, t_k) + w$ for some sequence $t_k \rightarrow \infty\}$. Then it is possible to show:

**Lemma.** $y \in \omega(x) \Rightarrow y \in \omega(z)$ for all $z$ sufficiently near the orbit $\theta(y)$ of $y$ and on the same side of $\theta(y)$ as $x$.

**Lemma.** $y \in \omega(x) \Rightarrow \omega(x) = \theta(y)$, $\omega(z)$ for all $z$ near $\theta(y)$ as above.

**Corollary.** $y \in \omega(x) \Rightarrow x \notin \omega(z)$ for any $z$.

Clearly, similar results hold for $\alpha$-sets (defined by $t_k \rightarrow \infty$).

The only other possibility is Case 2: $p(x) < \infty$ for $x$ arbitrarily close to $\theta(y)$. Then it is easy to show:

**Lemma.** For $x$ close enough to $\theta(y)$ and with $p(x) < \infty$ the orbit $\theta(x)$ is concentric with $\theta(y)$, and all such $\theta(x)$'s are concentric with each other.

Using the above results the configuration of orbits of $\phi$ can now be pieced together. Start with a periodic orbit $\theta(y)$. In Case 1 it follows that $(z \mid \alpha(z) \text{ or } \omega(z) = \theta(y)}$ is (topologically) an open annulus around $\theta(y)$, and its boundary must lie in $F(\phi)$ or contain a periodic orbit. Thus in Cases 1 or 2 we can build out from $\theta(y)$ producing concentric periodic orbits until (after a countable number of steps) we can go no further: then the set of points on or between these orbits is an open annulus with boundary in $F(\phi)$. Between two 'adjacent'