CHAPTER V. SIEGEL'S EISENSTEIN SERIES

Siegel's Eisenstein series are defined. It is shown that they may be derived from Selberg's Eisenstein series (representation with Selberg's zeta function) by computing residues of Selberg's zeta function as it was done in § 13. Hence the analytic continuation of Selberg's Eisenstein series gives us analytic continuation of Siegel's Eisenstein series. From the functional equations for Selberg's Eisenstein series one gets a functional equation for Siegel's Eisenstein series.

§ 19. SIEGEL'S EISENSTEIN SERIES

Analytic continuation and the functional equation for Siegel's Eisenstein series is obtained.

Like in (1054) define Siegel's Eisenstein series by

$$E(n,r,Z,u) = \sum_{M \in \Gamma(n) \setminus \Gamma(n)} (\det M|Z|)^{-2r} (\det Y_M)^{u-r}.$$  \hspace{1cm} (1169)

Obviously

$$E(n,r,M(Z),u) = (\det M|Z|)^{2r} E(n,r,Z,u) \quad (M \in \Gamma(n)).$$  \hspace{1cm} (1170)

Set

$$s^\nu(u) = u - \frac{m+1}{4} - \frac{m}{2}(w-\nu) \quad (\nu = 1, \ldots, w),$$  \hspace{1cm} (1171)

$$s(u) = (s^1, \ldots, s^w).$$  \hspace{1cm} (1172)

Then from (964), (1122) we deduce

$$s^m(u) = u - \frac{n+1}{4}.$$  \hspace{1cm} (1173)

An easy computation shows
THEOREM 113: Let axiom A be true and set
\[
B(m,w,r,u) = w^{-1} \prod_{\nu=0}^{w-1} K(m,r,u - \frac{\nu}{2}) (\prod_{1 \leq u \leq v \leq w-1} C(m,2u - \frac{1+m(u+v)}{2}))
\]
Then
\[
R(m,w,r,Z,u) = B(m,w,r,u)E(n,r,Z,u).
\]

Then
\[
R(m,w,r,Z,u) = \gamma(Det Y)^{-2r} p(m,w,r,Z,s(u))
\]
with some constant \( \gamma \). The function \( R(m,w,r,Z,u) \) is holomorphic for \( u \in \mathbb{C} \) and it satisfies the functional equation
\[
R(m,w,r,Z,u) = R(m,w,r,Z,\frac{n+1}{2} - u).
\]

PROOF: Let \( \gamma \) be a \( \gamma \)-domain. For large enough \( \text{Re} \, u \) we have
\[
s^{(m)}(u) \in \gamma.
\]
Hence theorems 87, 108 and formulas (1116), (1171), (1172), (1173), (1175), (1176) give (1177).

From theorem 111 it follows that \( R(m,w,r,Z,u) \) is holomorphic for \( u \in \mathbb{C} \). By theorem 111 the function \( P(m,w,r,Z,s) \) is invariant under \( \Delta(w) \). Hence
\[
P(m,w,r,Z,s) = P(m,w,r,Z,s).
\]
From (1174), (1180) we get (1178). Theorem 113 is proved.