ON THE NUMBER OF STRICTLY BALANCED SUBGRAPHS OF A RANDOM GRAPH

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SUMMARY

In this paper we consider the asymptotic distribution of the number of strictly balanced subgraphs of a random graph. We prove that, under certain conditions imposed on the probability of occurrence of edges, this number has the Poisson or the standard normal distribution. It generalizes several results dealing with subgraphs of various kinds of random graph.

Let $B_{r,a}$ denote an arbitrary non-empty class of labelled strictly balanced graphs on $r$ ($r \geq 2$) vertices and $a$ ($a \geq r - 1$) edges. An $(r,a)$-graph $G$ is strictly balanced if for every proper subgraph $H$ of the graph $G$, i.e., $H \subseteq G$, $|E(H)|/|V(H)| < a/r$ where $V(H)$, $E(H)$ are the vertex-set and the edge-set of $H$. For example $k$-trees (in particular trees and cliques), complete $k$-partite graphs, $k$-regular connected graphs and $k$-cubes are strictly balanced. Let $X_n$ be a random graph obtained from a complete labelled graph $K_n$ on $n$ vertices by a random and independent deletion of its edges with the same probability $p = p(n) \in (0,1)$. Suppose that $X_n = X_n(B_{r,a})$ is the number of subgraphs of $X_n$ such that each is isomorphic with some element of $B_{r,a}$.

**Theorem 1.** If

\begin{equation}
\lim_{n \to \infty} p(n)n^{r/a} = \rho \in (0,\infty)
\end{equation}

then

\[ \lim_{n \to \infty} \operatorname{Prob}(X_n = i) = \frac{\lambda^i}{i!} e^{-\lambda}, \quad \lambda = \rho \epsilon, \quad \epsilon > 0, 1, 2, \ldots, \]
where

\( \lambda = \frac{1}{n!} \rho m(B, r, a) \),

whereas \( m(B, r, a) \) denotes the number of all labelled \((r, a)\) - graphs isomorphic with some element of \( B, r, a \).

To prove the above theorem we need the following lemma.

**Lemma.** Suppose that \( G_1, G_2, ..., G_t \) \((t \geq 2)\), are pairwise different graphs isomorphic with some elements of \( B, r, a \) such that not all are pairwise vertex-disjoint. If \( t \) the graph \( \bigcup_{i=1}^{t} G_i \) has \( v_t \) vertices and \( e_t \) edges, then

\[
\frac{v_t}{e_t} \leq \frac{r}{a} - \varepsilon(t),
\]

where

\[
0 < \varepsilon(t) = \min_{0 \leq h \leq \tau - 1} \varepsilon_h(t),
\]

\[
e_0(t) = \frac{\tau}{a[(t-1)a+1]}, \quad \varepsilon_h(t) = \frac{a[r-h]-r-h}{a(ta-l_{p-h}-h)}, \quad l = \max_{G \in B, r, a} \max_{H \subseteq G} |E(H)|, \quad s = \max_{G \in B, r, a} |V(H)| = s
\]

\( h, s = 1, ..., \tau - 1. \)

**Proof** (by induction on \( t \)). First, note that \( \varepsilon(t) > 0 \) by the assumption that all elements of \( B, r, a \) are strictly balanced, moreover \( \varepsilon(t) \) is a decreasing function of \( t \). Let \( h = h(t) = \left| V(G_i) \setminus \bigcup_{i=1}^{t-1} V(G_i) \right| \). Consider the case \( t = 2 \) and assume that \( h = 0 \).

Then \( v_2 = r, e_2 = a+1 \) so \( \frac{v_2}{e_2} \leq \frac{r}{a+1} \leq \frac{r}{a} - \varepsilon \). The last inequality holds for \( \varepsilon \leq \varepsilon_0(2) \).

Suppose now that \( 1 \leq h \leq \tau - 1 \). Then clearly \( \frac{v_2}{e_2} \leq \frac{r+h}{2a-l_{p-h}} \leq \frac{r}{a} - \varepsilon \) for \( \varepsilon \leq \varepsilon_h(2) \).

Assume that our lemma is true for \( t-1 \) and let us choose \( t \) elements of \( B, r, a \) fulfilling the assumptions of the lemma. There exists a graph, say \( G_t \), such that \( h \leq r-1 \). Consider now the following two cases.

Case 1. \( G_1, G_2, ..., G_{t-1} \) are not pairwise vertex-disjoint. Now, if \( h = 0 \) then

\[
\frac{v_t}{e_t} \leq \frac{v_{t-1}}{e_{t-1}} \leq \frac{r}{a} - \varepsilon(t-1) \leq \frac{r}{a} - \varepsilon(t),
\]

whereas for \( 1 \leq h \leq r-1 \)