GENERALIZED HOMOMORPHISMS BETWEEN C*-ALGEBRAS AND KK-THEORY

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In the category of C*-algebras, the natural morphisms are, of course, the C*-algebra homomorphisms, i.e. the algebra homomorphisms which commute with the involution (such homomorphisms are automatically norm-decreasing). One disadvantage of this notion of a morphism is that, given two C*-algebras A and B, there are in general very few homomorphisms from A to B; in fact, typically, often the only possible homomorphism is the 0-map. For the study of homotopy and topological properties of C*-algebras one would like a more flexible class of morphisms. In practice, very different looking C*-algebras can be homotopy equivalent, in a generalized sense, but the equivalence is induced, rather than by a homomorphism, by what we shall call a quasihomomorphism. A quasihomomorphism from A to B is, essentially, a pair of homomorphisms $\phi,\bar{\phi}$ from A to E, where E is a C*-algebra containing an ideal J with $J \subset B$, such that $\phi(x) - \bar{\phi}(x) \in J$ for all $x \in A$. With this definition, there are, in general, many quasihomomorphisms from A to $K \otimes B$, where K is the algebra of compact operators on a Hilbert space of countably infinite dimension. Another advantage is, that the "negative" $(\bar{\phi},\phi)$ of a quasihomomorphism $(\phi,\bar{\phi})$ is again a quasihomomorphism.

Our aim, in this article, is to develop some of the fundamental properties of Kasparov's KK-theory [4] on the basis of the notion of a quasihomomorphism. The group $KK(A,B)$ can be defined as the set of all homotopy classes of quasihomomorphisms from A to $K\otimes B$. We then construct the product $KK(A_0,A_1) \times KK(A_1,A_2) \to KK(A_0,A_2)$ which is the heart of Kasparov's theory (and generalizes the composition of homotopy classes of homomorphisms). Our definitions allow us to avoid many of the technicalities that are part of Kasparov's work, like graded algebras, Hilbert modules or stabilization. We even, as a variation, replace Kasparov's main technical theorem by a related theorem of Pedersen, in the construction of the product. The associativity of the product, a point of great importance, which is rather mysterious in [4], and still somewhat unnatural in [6], is nearly automatic in our approach. Additional

*partially supported by NSF

minor features are, that we do not use the rather unintuitive stabilization theorem at all, and that we don't have to assume the existence of strictly positive elements in A_1 and A_2 for the construction of the product.

For certain applications, our frame for KK-theory may prove less clumsy and easier to handle than the one of Kasparov. Our approach emphasizes the nature of elements of KK as generalized homomorphisms, cf. also [2], while in Kasparov's work an element of KK appears rather as a generalized elliptic operator. Thus, in situations as in [1] where one handles elements of KK arising naturally from such operators, it will, presumably, be preferable to work with Kasparov's definitions and with the product as defined in [4] or the modification in [1].

This article is meant to be read in conjunction with the expository article [3] which gives an outline of K-theory for C*-algebras and contains a discussion and some applications of the present ideas.

I am very grateful to Daniel Kastler and his colleagues at the CPT-CNRS, Marseille-Luminy for their hospitality during a stay which gave me the leisure to get acquainted with Kasparov's theory.

1. Preliminaries.

An important tool in C*-algebra theory is the so-called multiplier algebra $M(A)$ for a C*-algebra A. A multiplier (or double centralizer) of A is a pair (T_1,T_2) where T_1,T_2 are (necessarily bounded) linear maps from A to A such that $T_1(xy) = xT_1(y)$, $T_2(xy) = T_2(x)y$ and $T_1(x)y = xT_2(y)$ for all $x,y \in A$. The set of all multipliers of A, with the obvious addition, the product $(T_1,T_2)(T_1',T_2') = (T_1 \cdot T_1', T_2' \cdot T_2)$ the involution $(T_1,T_2)^* = (T_2^*,T_1^*)$ where $T_i^*(x) = (T_i(x^*))^*$, $i = 1,2$, and the operator norm, forms a C*-algebra. This C*-algebra is called the multiplier algebra of A and denoted $M(A)$. If A is faithfully represented on a Hilbert space H, then $M(A)$ is isomorphic to the algebra $\underline{A} = \{x \in L(H) \mid xa, ax \in A$ for all $a \in A\}$ via the isomorphism $\underline{A} \ni x \to (T_1,T_2) \in M(A)$ with $T_1(a) = ax$, $T_2(a) = xa$ for $a \in A$; cf. [5,3.12]. The C*-algebra A is contained in $M(A)$ as an ideal (all our ideals are closed). We have the following basic fact.

1.1 <u>Proposition</u>. Let J be an ideal in a C*-algebra E and $\phi: J \to B$ a homomorphism into some C*-algebra B. Then ϕ extends uniquely to a homomorphism $\phi': E \to M(\phi(J))$. If J is essential (i.e. $xJ = \{0\}$ implies $x = 0$ for $x \in E$) and ϕ is injective, then ϕ' is injective.