§ 1. Smart and coarse theorems.

There are two problems, namely desingularization and simplification of a boundary and for each of them two levels of precision in the corresponding theorems which I will call coarse and smart.

Coarse desingularization theorem. Let $X$ be a scheme (or a complex analytic space). Assume that $X$ is reduced. There exists a proper morphism $\pi: X' \longrightarrow X$, such that the induced morphism $\pi^{-1}(X_{\text{reg}}) \longrightarrow X_{\text{reg}}$ is an isomorphism, where $X_{\text{reg}}$ is the set of points where the local ring $O_{X,x}$ is regular, and $X'$ is regular.

This is a theorem in the complex analytic case or for $X$ excellent of characteristic zero or for $X$ excellent and $\dim X \leq 2$.

Coarse Simplification of boundary. Let $Z$ be a regular scheme (or smooth complex analytic space), let $Y$ be a closed subset of $Z$. There exists a proper and birational morphism $\pi: Z' \longrightarrow Z$ such that

1. $Z'$ is regular
2. $\pi^{-1}(Y)$ is a normal crossing divisor,

(for short we say that $\pi^{-1}(Y)$ is a d.n.c.) and each irreducible of $\pi^{-1}(Y)$ is regular.

Of course if $Y$ is regular we achieve that by blowing up $Y$.

There are various smart versions of these theorems and it is not my intention to discuss them. I will only give one of them.
Smart desingularization theorem. Let $X$ be an excellent scheme of characteristic zero.

There exists a sequence

$$
X = X_0 \leftarrow X_1 \leftarrow X_2 \ldots \leftarrow X_{N-1} \leftarrow X_N
$$

such that

(i) $Y_{i+1}$ is regular, closed in $X_i$, and contained in the singular locus of $X_i$, $0 \leq i \leq N-1$.

(ii) $X_{i+1}$ is the blowing up of $X_i$ with center $Y_{i+1}$.

(iii) $X_N$ is regular.

(iv) if $E_i$ is the inverse image of $Y_i$ in $X_i$, $1 \leq i \leq N$, then $\bigcup_{1 \leq i \leq N} E_i$ is a divisor with normal crossings.

Here the important condition is (i), because if you start with some embedding of $X$ as a closed subscheme of a regular scheme $Z$, then by letting $Z_0 = Z$, and $Z_{i+1} = $ blowing up of $Z_i$ with center $Y_{i+1}$, you get that each $X_i$ is closed in a regular $Z_i$. In other words if $X_i$ is embedded in a regular $Z$, then the desingularization $X_N$ is embedded in a regular $Z_N$.

Remark 1. If $X$ is embedded as a hypersurface in a regular $Z$, then the coarse simplification of boundary will produce $\pi: Z' \rightarrow Z$ such that $\pi^{-1}(X)$ is a d.n.c. Hence, if $X$ is reduced and irreducible we will get a birational desingularization $X' \rightarrow X$, by taking for $X'$ the component of $\pi^{-1}(X)$ whose image is a divisor in $Z$. This will only be a coarse desingularization of $X$, unless we have a smart simplification of boundary, by which I mean some condition implying condition (i) of the smart desingularization theorem.

Remark 2. On the other hand if $X$ is a curve in $\mathbb{P}_3 = Z$, then simplification of boundary will not give a desingularization of $X$ since the strict transform of $X$ is going to be empty because one will