Let $G$ be a unimodular locally compact group, and let $\pi$ be an irreducible unitary representation of $G$ on a Hilbert space $H_\pi = H$. A matrix coefficient of $\pi$ is a function of the form

$$f_{v,w}(x) = \langle \pi(x)v, w \rangle,$$

where $v, w \in H$. (In what follows, we adopt the convention that $v$ and $w$ are never 0 unless the case $v = 0$ (or $w = 0$) is trivially needed to make a result nontrivially true.) The function $f_{v,w}$ is obviously constant on cosets of $G_\pi = \text{Ker} \pi$.

We say that $\pi$ has $L^p$ matrix coefficients $(1 \leq p < \infty)$ if there are (nonzero!) vectors $v, w$ such that $f_{v,w} \in L^p(G/G_\pi)$. Of course, all matrix coefficients are in $L^\infty$. It is easy to check that if $f_{v,w} \in L^p(G/G_\pi)$, then so are $f_\pi(x)v, w$ and $f_{v,\pi(x)w}$ for every $x \in G$; moreover, if $v$ is fixed, then $\{w : f_{v,w} \in L^p(G/G_\pi)\}$ and $\{w : f_{w,v} \in L^p(G/G_\pi)\}$ are subspaces of $H$. Hence if $\pi$ has $L^p$ matrix coefficients, then there are dense subspaces $V, W$ of $H$ such that $f_{v,w} \in L^p(G/G_\pi)$ for all $v \in V$ and all $w \in W$. For $p = 2$, more is true: if one matrix coefficient is in $L^2$ and if $G/G_\pi$ is unimodular, then all matrix coefficients are $L^2$. (See, e.g., [3], p. 278.)

Now suppose that $G$ is a nilpotent, connected, simply connected Lie group with Lie algebra $\mathfrak{g}$. There seem to be two main papers in the literature dealing with matrix coefficients of $G$: Moore-Wolf ([6]) and Howe-Moore ([5]). It is not surprising, therefore, that the work I am going to describe is the result of a collaboration with Calvin Moore. (The work is still in progress; this is a preliminary report.)

The paper [6] is devoted to a study of square integrable representations (the case $p = 2$). The basic theorem is:
Theorem A. Let $\pi \in G^*$, and let $0_\pi$ be the Kirillov orbit in $g^*$ corresponding to $\pi$. Then $\pi$ has $L^2$ matrix coefficients iff $0_\pi$ is flat (a coset of a subspace of $g^*$). Equivalently: $\pi$ has $L^2$ matrix coefficients iff for any $\ell_0 \in 0_\pi$, $0_\pi = \ell_0 + R_{\ell_0}^\perp$. Here, $R_{\ell_0} = \text{radical of } \ell_0 = \{X \in g : \ell_0([X,Y]) = 0 \text{ for all } Y \in g\}$.

One obvious question is whether these representations have matrix coefficients that are better than $L^2$. We have the following answer:

Theorem 1. Suppose that $\pi$ has $L^2$ matrix coefficients (here and below, $\pi$ is an irreducible unitary representation of the connected, simply connected nilpotent Lie group $G$). Let $\pi$ act on $H$, and let $v, w$ be vectors for $\pi$. Then $f_{v,w}$ is a Schwartz class function on $G/G_{\pi}$.

Theorem 1 might be interpreted as saying that the matrix coefficients of $\pi$ are in $L^\infty$ for all $\ell > 0$. Incidentally, there is a similar theorem for $p$-adic nilpotent Lie groups; see [7].

Now suppose that $\pi$ is not square integrable. Let $G_{\pi}^\sim$ be the projective kernel of $\pi$ (i.e., $x \in G_{\pi}^\sim \Rightarrow \pi(x)$ is a multiple of $I$); then $|f_{v,w}|$ is constant on cosets of $G_{\pi}^\sim$, and we can ask about the behavior of $|f_{v,w}|$ on $G/G_{\pi}^\sim$. (In the groups we care about, $G_{\pi}/G_{\pi} \cong T$, so that the distinction between $G_{\pi}^\sim$ and $G_{\pi}$ is not too important.)

In [4], the following result was proved:

Theorem B. For all $v, w \in H$, $|f_{v,w}|$ vanishes at $v, w$ on $G/G_{\pi}^\sim$.

We can improve on this somewhat. The group $G_{\pi}^\sim$ is connected; let its Lie algebra be $\bar{g}_{\pi}^\sim$. Let $\bar{G}_{\pi} = G/G_{\pi}^\sim$, $\bar{g}_{\pi} = g/\bar{g}_{\pi}^\sim$. Choose a Euclidean norm, $|\cdot|$, on $\bar{g}_{\pi}$, and lift it back to $\bar{G}_{\pi}$ via exp.

Theorem 2. There are vectors $v, w$, and constants $C, \gamma > 0$ such that on $\bar{G}_{\pi}$,

$$|f_{v,w}(x)| \leq C(1+|x|)^{-\gamma}.$$  

To see why Theorem 1 is true, look at the 3-dimensional Heisenberg