Abstract. The fact that the Taylor series expansion of an analytic function converges inside the largest disk of analyticity of the function and diverges outside the disk is generalized to interpolation with rational functions where the points of interpolation are chosen in a very general way.

0. Introduction

Let \( n \) and \( \nu \) be non-negative integers, let \( \beta_{j_\nu}, 1 \leq j \leq n+\nu+1 \), be given points in the complex plane \( \mathbb{C} \), not necessarily distinct, and let \( f \) be a function which is analytic at least at the interpolation points. Then there exist polynomials \( P_{n_\nu} \) and \( Q_{n_\nu}, Q_{n_\nu} \neq 0 \), of degree at most \( n \) and \( \nu \), respectively, such that \( fQ_{n_\nu} - P_{n_\nu} \) is zero at \( \beta_{j_\nu} \) for all \( j \); if two interpolation points coincide, we require the corresponding zero to have multiplicity two, etc. It is also easy to prove that \( R_{n_\nu} = P_{n_\nu}/Q_{n_\nu} \) is uniquely determined by \( f \) and \( \beta_{j_\nu}, 1 \leq j \leq n+\nu+1 \); \( R_{n_\nu} \) is the multipoint Padé approximant of type \((n,\nu)\) of \( f \) and \( \{\beta_{j_\nu}\} \), and when \( \beta_{j_\nu} = 0 \), we get the classical Padé approximant of type \((n,\nu)\) of \( f \). When \( \nu = 0 \), \( R_{n_\nu} \) is a general interpolation polynomial of \( f \) which, for \( \beta_{j_\nu} = 0 \), becomes the Taylor polynomial of \( f \).

We have the basic fact that the Taylor polynomial of \( f \) of degree \( n \) around zero converges, as \( n \to \infty \), inside the largest disk around zero where \( f \) is analytic, that \( f \) has a singularity on the boundary of that disk and that the Taylor series diverges outside the disk. For the Padé approximants we have an analogous result by de Montessus de Ballore saying that if \( f \) is analytic at zero and meromorphic with
ν poles in the open disk $E(ρ)$ around zero with radius $ρ$, then the Padé approximant $R_{nν}$ of type $(n,ν)$ of $f$ converges, as $n$ tends to infinity, to $f$ in $E(ρ)$ except at the poles of $f$. Furthermore, if $E(ρ')$ is the largest such disk, then $f$ has a singularity on the boundary of $E(ρ')$ and $R_{nν}$ diverges in $|z| > ρ'$. We refer to [4], §2 or [1], or to §§3-4 in this paper for the proof of these results; a related divergence result is in [6]. For the multipoint Padé approximant $R_{nν}$ we have an analogous convergence result (Theorem 1 in §1 and [10], Theorem 1) essentially going back to Walsh [11] in the polynomial case ($v=0$), and to Saff [2] and Warner [13] in the case $ν > 0$. The purpose of this paper is to prove a corresponding divergence theorem (Theorem 3 in §1) for the multipoint Padé approximant in the case when the interpolation points $β_{jν}$ are independent of $n$; a related but different divergence theorem was proved in [9]. In order to prove this divergence result we need a further fact on the convergence behaviour (Theorem 2 in §1). The proofs of Theorem 2 and 3 given below are generalizations of the corresponding proofs in the Padé case ($β_{jν} = 0$) given in [4]. In §1 the results are formulated and some further references are given. After some preparation in §2 the theorems are proved in §3 and §4.

1. Definitions and results

The definition of the multipoint Padé approximant $R_{nν} = P_{nν}/Q_{nν}$ of type $(n,ν)$ of $f$ and $β_{jν}$, $1 ≤ j ≤ n+ν+1$, may also be stated in the following way by using the auxiliary polynomial

$$w_{nν}(z) = \prod_{j=1}^{n+ν+1} (z - β_{jν}).$$

(1.1)

Determine $P_{nν}$ and $Q_{nν}$, $Q_{nν} ≠ 0$, as polynomials of degree at most $n$ and $ν$, respectively, so that

$$(fQ_{nν} - P_{nν})/w_{nν}$$

is analytic at $β_{jν}$, $1 ≤ j ≤ n+ν+1$. (1.2)

We assume throughout that the interpolation points $β_{jν}$ all belong to a fixed compact subset $E$ of $C$ and that $f$ is analytic at the interpolation points. We let $ν > 0$ be fixed and define the associated measure $ν_n = ν_{nν}$ to $β_{jν}$, $1 ≤ j ≤ n+ν+1$, as the probability measure on $E$ which distributes the point mass $1/(n+ν+1)$ at each of the points $β_{jν}$, $1 ≤ j ≤ n+ν+1$. We shall assume that $ν_n$, $n=1,2,...,