PARTIAL DIFFERENTIAL APPROXIMANTS
AND THE ELUCIDATION OF MULTISINGULARITIES

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Abstract. A partial differential approximant, or PDA, \( F(x,y) \), can accurately approximate a two-variable function, \( f(x,y) \), on the basis of its power series expansion even near a multisingular point where the function is intrinsically nonanalytic in both variables. This brief review argues that multisingularities occur frequently in two-variable functions arising in practical situations. Partial differential approximants are defined and it is shown why they can approximate multisingularities. The invariance of PDAs under a change of variables is discussed and new results are presented concerning functions exactly representable by PDAs. Finally, several applications of PDAs are mentioned.

1. Introduction

Any effective approximant to any function must be capable of exhibiting the important behavior displayed by the function being approximated. This known functional behavior, which must be imitated by the approximant, typically consists of two parts: first, there is the behavior near the origin, which we suppose given by a finite number of low order terms in the formal power series expansion

\[
f(x,y) = \sum_{i,i'=0}^{\infty} f_{i,i'} x^i y^{i'} .
\]

Second, in any practical calculation there is always some idea of how
the function will behave away from the origin. For example, one might know the behavior of \( f(x,0) \) as \( x \to \infty \), or one might suspect that \( f(x,y) \) has some particular singularity structure. In the following Section we first briefly review how this situation may be handled effectively in the single-variable case. We then argue heuristically that a particular but rather general singularity structure, the so-called scaling form, will frequently be found in two-variable functions arising in various different practical contexts. Lastly, a partial differential approximant is defined as the solution of an appropriate first order two-variable boundary value problem constructed from the coefficients \( f_{i,i'} \) (and possibly embodying other known information).

The third section concerns the theory of partial differential approximants. After establishing that a partial differential approximant can, in fact, fulfill the two requirements for an effective approximant mentioned above, the invariance properties of partial differential approximants under a change of variable, such as an Euler transformation, are examined. This section also includes a short discussion of some practical techniques which permit significant information to be gleaned from an approximant with relatively little numerical labor.

Finally, Section 4 surveys some of the problems to which partial differential approximants have been applied. This section is short but the interested reader may consult other recent reviews which emphasize applications [12,26].

2. Background Considerations

Many functions which arise in physics, engineering, combinatorics, and related fields exhibit branch point singularities of the form

\[
f(x) \approx A(x)(x-x_c)^{-\gamma} + B(x), \quad \text{as} \quad x \to x_c,
\]

where \( A(x) \) and \( B(x) \) are functions analytic at \( x=x_c \). It has long been known that when \( \gamma > 0 \) and the "background function", \( B(x) \), is small, such functions can be approximated effectively by Baker's Dlog Padé technique [1]. More recently [11,19], non-negligible background functions have been treated successfully using the method of "inhomogeneous differential approximants" (or "integral curve approximants") in which \( f(x) \) is approximated by the solution, \( F(x) \), of the ordinary