ON ERGODIC THEORY AND TRUNCATED LIMITS IN BANACH LATTICES

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We obtain here some ergodic results for positive operators acting on a weakly sequentially complete Banach lattice $E$. The proofs are by a truncation method introduced in [1]. The first section discusses general properties of "truncated limits". In the second section we obtain necessary and sufficient conditions for the existence of invariant weak units (strictly positive fixed points, in a different terminology). One such condition is that truncated limits of band projections of subsequences of averages do not vanish. In third section we prove the strong convergence of $(A_n f) \wedge \phi$, where $A_n$ are Cesàro averages of a positive contraction, $f \in E_+$, and $\phi$ is an arbitrary invariant positive element. Without an additional assumption on the Banach lattice, the truncated limit of $A_n f$ need not exist.

1. Properties of Truncated Limits.

Let $E$ be a Banach lattice. Our terminology will be that of the book Lindenstrauss-Tzafriri [17], to which we will refer by [LT]. In the present article we will make only the following two assumptions (A) and (B):

(A) There is an element $u$ in $E_+$ such that if $f$ is in $E_+$ and if $u \wedge f = 0$, then $f = 0$. Such an element $u$ is called a weak unit.

(B) Every norm-bounded increasing sequence in $E$ has a strong limit.

Assumptions equivalent with (B) are: (B') $E$ is weakly sequentially complete, and also: (B'') $E$ contains no isomorphic copy of $c_0$ ([LT], p. 34). (B) implies that $E$ is order-continuous. Therefore, the assumption (A) that there is a weak unit is not a loss of generality if $E$ is separable ([LT], p. 9).

Since the condition (B) implies order-continuity, one has

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1.1. Every order interval $[f, g] = \{h : t < h < g\}$ is weakly compact ([LT], p. 28).

Norm convergence will be simply called convergence and denoted by $\rightarrow$. Weak convergence is $\rightarrow^w$, and order convergence for monotone sequences is denoted by $\rightarrow$ and $\rightarrow^o$.

1.2. Let $\phi \in E_+$. Then there is a linear bounded operator $P = P_\phi : E \rightarrow E$ such that $Pf = \lim f \land (n\phi)$ for each $f \in E_+$ (limit in strong topology). Then $P$ is a band projection (on $\phi$), implying that $\pi P \pi < 1$, $P^2 = P$, and $P E$ is a sub Banach lattice of $E$. $Q = 1 - P$ is another band projection.

1.3. A band projection $P_u$ on a weak unit $u$ is the identity, i.e., if $f \in E_+$, then $f \land nu = f$.

In other terms, a weak unit is necessarily a quasi-interior point ([20], p. 96) or a topological unit.

1.4. There exists a strictly positive element $U$ in $E^*_+$, i.e., a $U$ such that $f = 0$ if $U|f| = 0$ ([LT], p. 25; if $E$ is separable, this is very easy to prove).

1.5. If $f_n$ in $E_+$ is such that $f_n \rightarrow^w 0$ and $\sup f_n \in E$, then $f_n \rightarrow 0$.

Proof. If the conclusion fails, then passing to subsequences we can assume that $\|f_n\| > \varepsilon > 0$ for all $n$ and $\sum U f_n < \infty$ for a strictly positive $U$ in $E^*_+$. Let $g_n = \sum_{k=1}^{n} f_k$, $g = \land g_n$. Then $g_n \rightarrow^w g$, hence $\|U g_n - U g\| < \varepsilon$. If $U g < \sum_{k=1}^{n} U f_k$ implies that $U g = 0$, hence $g = 0$. This contradicts $\|U g\| = \lim\|g_n\| > \varepsilon$.

1.6. Definition of truncated limits.

Let $f_n \in E_+$, $\phi \in E_+$. Then $\text{TL} f_n = \phi$ (the truncated limit of $f_n$ is $\phi$) means that for a weak unit $u$, $\lim (f_n \land ku) = \phi_k$ exists for each $k$, and $\phi_k \rightarrow^o \phi$. 
