THE JENKINS' TYPE INEQUALITY FOR BAZILEVIČ FUNCTIONS

Wiesława Drozda (Olsztyn),
Anna Szynal and Jan Szynal (Lublin)

1. Let \( S \) denote the class of holomorphic and univalent functions \( f \) in the unit disk \( K = \{ z : |z| < 1 \} \) which have the form

\[
(1) \quad f(z) = z + a_2 z^2 + \ldots, \quad z \in K.
\]

By \( B(\alpha, \beta; g, h_a) \) we denote the class of Bazilevič functions. Namely, we say that a holomorphic function \( f \) of the form (1) is in \( B(\alpha, \beta; g, h_a) \) if it has the representation

\[
(2) \quad f(z) = \left\{ \frac{\alpha + i\beta}{1 + i\alpha} \right\} \int_0^z g^\alpha(s) h_a(s) s^{i\beta-1} \, ds, \quad z \in K.
\]

The parameters \( \alpha \) and \( \beta \) satisfy the conditions: \( \alpha > 0, \beta \in (-\infty, +\infty) \), whereas \( g \) is a starlike and univalent function in \( K \) of the form (1) and \( h_a = h_a(z) = 1 + a_1 z + 2h_2 z^2 + \ldots \) (\( a_1 \) is real) is holomorphic in \( K \) and satisfies the condition \( \Re h_a(z) > 0, \ z \in K \).

Bazilevič [1] (see also Pommerenke [6]) showed that the functions \( f \) given by (2) are univalent in \( K \).

From (2) one can get the differential equation for \( f \in B(\alpha, \beta; g, h_a) \). After some calculations we obtain from (2):

\[
(3) \quad 1 + \frac{zf'(z)}{f(z)} + (\alpha - 1 + i\beta) \frac{zf''(z)}{f(z)} = 1 + \frac{zh_a'(z)}{h_a(z)} + \alpha \frac{g(z)}{g'(z)}.
\]

The class \( B(\alpha, \beta; g, h_a) \) with particular choices of \( \alpha, \beta, g \) and \( h_a \) yields several classes of univalent functions.
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We quote below some of them:

(a) $B(1,0;g,h_a) = L$ - the class of close-to-convex functions,
(b) $B(1,tg_1;g,1) = S^g_r (|Y|<\frac{1}{2}r)$ - the class of spiral-convex functions [10],
(c) $S^C = S^C -$ the class of convex functions,
(d) $\lim_{\alpha \to 0} B(\alpha, \alpha tg_1; g,1) = S^g_r (|Y|<\frac{1}{2}r)$ - the class of spiral-like functions,
(e) $S^*_o = S^*_o -$ the class of starlike functions,
(f) $B(\frac{1}{g},0;g,1) = M_\alpha -$ the class of $\alpha$-convex functions in the sense of Mocanu.

The coefficient problem for the class $S$ has been attacked for many years and the results concerning that problem are well known.

However, as for the class of Bazilevič functions, very little is known about coefficients and even for initial coefficients there are only partial results [2] and [4].

In the estimates of some functionals concerning $a_2$ and $a_3$ for the class $S$ the important role plays so-called Jenkins' inequality.

This inequality has the form

$$\Re \left\{ (a_3 - a_2^2) - \lambda a_2 \right\} \leq 1 + \frac{8\lambda^2}{(1+\lambda^2) \log \frac{|\lambda|}{N}},$$

where $\lambda \in [-4,4]$ and $f$ is an arbitrary function in $S$. (For a short proof of (4) we refer to [3] or [8]).

The extension of (4) for complex $\lambda$ and bounded univalent functions is much more complicated but, as it was pointed out in [9], leads also to the determination of the coefficient region $(a_2,a_3)$ within the considered classes.

In this paper we are going to prove the Jenkins' type inequality for the class $B(\alpha,\beta;g,h_a)$ and show some applications of it.

The obtained results will have quite complicated forms, however in some special cases it will be possible to find concise and clear estimates.

In what follows we denote by $B^r(\alpha,\beta;g,h_a)$ the subclass of Bazilevič functions consisting of functions with real coefficients. In the same way we indexed all the other classes which appear in the paper and which consist of functions with real coefficients.

2. In order to get the Jenkins' type inequality for the class