IV. THE SOLUTION SPACE OF AN OPERATOR
AND AUTOMATIC WELL-POSEDNESS

Many physical problems may be modeled as an abstract Cauchy problem (0.1), where \( A \) is an operator on a locally convex space. It is well known that, in order for this model to have any practical value, it is not enough to have plenty of solutions; (0.1) should also be well-posed; informally, this means that small changes in \( x \), the initial data (corresponding to small errors in measurement) should yield small changes in \( u(t,x) \).

Although there is sometimes uncertainty about what precisely constitutes well-posedness, one condition that most would agree implies well-posedness is having \( A \) generate a strongly continuous semigroup. This implies that a sequence of solutions \( u(t,x_n) \) converges to \( u(t,x) \), uniformly on compact subsets of \([0,\infty)\), whenever the initial data \( x_n \) converge to \( x \).

It is often the case that \( A \) does not generate a strongly continuous semigroup. Consider the following list, which includes what are perhaps the most well-known partial differential equations: heat equation, Schrödinger equation, wave equation, Cauchy problem for the Laplace equation, backwards heat equation, all on \( L^p(\Omega) \), for appropriate \( \Omega \subseteq \mathbb{R}^n \), with appropriate boundary conditions. For each equation, when written as an abstract Cauchy problem (0.1), let us ask whether the operator \( A \) that appears generates a strongly continuous semigroup, for \( 1 \leq p < \infty \). The answers are, respectively, "yes," "sometimes," "sometimes," "no" and "no." Yet in all these cases, a unique solution exists, for all initial data in a dense set.

The class of operators that generate \( C \)-regularized semigroups is much larger than the class of operators that generate strongly continuous semigroups. In all the examples in the previous paragraph, we shall see in later chapters that \( A \) generates a \( C \)-regularized semigroup. The choice of \( C \) measures how ill-posed the problem is. When \( C \) is chosen to be \((\lambda - A)^{-n}\), for some \( n \in \mathbb{N} \), then \( i\Delta \), on \( L^p(\mathbb{R}^k) \), for \( 1 \leq p \leq \infty, p \neq 2 \), may be shown to generate a \( C \)-regularized semigroup, but not a strongly continuous semigroup (see Chapter XI). This yields solutions of the Schrödinger equation, for all initial data in the domain of \( \Delta^{n+1} \). Much "worse" operators, corresponding to what are traditionally referred to as ill posed or improperly posed problems, generate \( C \)-regularized semigroups. For example, if \( A \equiv -\Delta \), so that (0.1) becomes the backwards heat equation, then \( A \) generates a \( C \)-regularized semigroup (see Chapter VIII). If \( A \equiv \begin{bmatrix} 0 & I \\ -\Delta & 0 \end{bmatrix} \), so that (0.1) becomes the Cauchy problem for the Laplace equation, then \( A \) generates a \( C \)-regularized semigroup (see Chapter IX).
When $\text{Im}(C)$ is dense, as is the case in the examples above, then (0.1) has a unique solution for all $x$ in a dense set. Thus, $C$-regularized semigroups may often be used to produce unique solutions for all initial data in a dense set. In this chapter and the next, we will address the question of whether the solutions are well-posed, in some sense.

In this chapter, we show that, if $A$ is closed and there exists a nonempty set of initial data for which the abstract Cauchy problem (0.1) has a unique mild solution, then there exists a Frechet space, $Z$, that contains all such initial data, on which $A$ generates a strongly continuous semigroup. This is saying that we can always make (0.1) well-posed, regardless of how ill-posed our original formulation was. And if one adopts the working hypothesis that all physically correct models are well-posed, this construction may be considered a way of automatically correcting one's first guess; the space on which $A$ acts is often chosen out of convenience.

As a corollary, we obtain a much shorter, easier proof, than currently exists, of the well-known relationship between the abstract Cauchy problem and strongly continuous semigroups (Corollaries 4.11 and 4.12). This is a fundamental result that has seen many proofs, all of them somewhat involved, even when $X$ is a Banach space.

More generally, we use the solution space to show that, for any bounded operator $C$, the existence of a unique solution of (0.1), for all initial data in the image of $C$, corresponds to $A$ having a mild $C$-existence family (see Chapter II). When $A$ and $C$ commute and $C$ is injective, this is a $C$-regularized semigroup (see Chapter III.)

The only objection to the solution space is that it is not practical, in general, to try to construct it explicitly. $C$-existence families and $C$-regularized semigroups provide a simple method of approximating the solution space and its topology, as follows.

We will write $Y \hookrightarrow X$ to mean that $Y$ is continuously embedded in $X$, that is, $Y \subseteq X$ and the identity map from $Y$ to $X$ is continuous.

We show that, when $A$ generates a $C$-regularized semigroup, then

$$[\text{Im}(C)] \hookrightarrow Z \hookrightarrow X,$$

and $A|_Z$, the restriction of $A$ to $Z$, generates a strongly continuous semigroup. If the regularized semigroup is exponentially bounded, then we shall see in the next chapter that we may choose a Banach space; in general, $Z$ is a Frechet space. If $\rho(A)$ is nonempty, the converse is also true. The norm on the solution space $Z$ can then be expressed in terms of the regularized semigroup: $\|x\|_Z \equiv \sup\{\|C^{-1}W(t)x\| : t \geq 0\}$, where $\{W(t)\}_{t \geq 0}$ is the $C$-regularized semigroup generated by $A$.

This chapter shows that, in a technical sense, at least if one is willing to make renormings, the concepts of existence family and strongly continuous