Chapter 7

Probability Metrics in the Space of Random Sets Distributions

7.1 Definitions of Probability Metrics.

In this chapter we discuss probability metrics in the space of random closed sets distributions. Probability metrics method and its applications to limit theorems were elaborated by Zolotarev (1986), Kalashnikov and Rachev (1988), Rachev (1991). This method is developed mostly for distributions of random variables. There are many examples of probability metrics for random variables and inequalities between these metrics.

The probability metrics method enables to prove limit theorems for the most convenient metric. Afterwards, estimates of the speed of convergence are reformulated for other metrics by the instrumentality of inequalities between metrics. Sometimes this method allows to drop the condition of the uniform smallness of summands in limit theorems, i.e. to prove "non-classical" versions of limit theorems.

The probability metric $m(\xi, \eta)$ is a numerical function on the space of distributions of random elements. It satisfies the following conditions:

1. $m(\xi, \eta) = 0$ implies $P\{\xi = \eta\} = 1.$
2. $m(\xi, \eta) = m(\eta, \xi).$
3. $m(\xi, \eta) \leq m(\xi, \zeta) + m(\zeta, \eta).$

In this section several probability metrics for random sets are defined. They enable to determine distances between random sets distributions. Later on their applications to limit theorems for unions are considered.

Since a random set is an $F$-valued random element, probability metrics for random sets can be defined by specializing general metrics for the case of random elements in the space $F$ furnished with $\sigma$-algebra $\sigma$ and the Hausdorff distance $\rho_H$.

In such a way the Levy-Prohorov metric can be defined, because its form does not depend essentially on the structure of the setting space. We can also define the metric $K_H$ as

$$K_H(X, Y) = \inf \{\varepsilon > 0: P\{\rho_H(X, Y) > \varepsilon\} < \varepsilon\},$$

where $X$ and $Y$ are random compact sets. It can be shown that $K$ metrizes the convergence of random compact sets in probability with respect to the Hausdorff
The analog of so-called "engineering" metric (see Zolotarev, 1986) is defined as
\[ I_H(X, Y) = \mathbb{E}_{\rho_H}(X, Y). \]

The enlisted metrics are composite, i.e. their values depend on the mutual distributions of \( X \) and \( Y \). It is well-known that simple metrics are more convenient, since they can be naturally applied to limit theorems. A probability metric is said to be simple if its values depend only on marginal distributions of random elements (random sets).

Many interesting simple metrics for random variables are defined by the corresponding densities or characteristic functions. Unfortunately, they cannot be reformulated for random sets directly, since the space \( \mathcal{F} \) of closed sets does not admit a group operation and there are not analogues of the Lebesgue measure and densities for \( \mathcal{F} \)-valued random elements (random sets).

Another approach is based on the notion of selector for random sets, see Wagner (1979). The random element \( \xi \) is said to be a selector of \( X \) if \( \xi \in S(X) \). If the random closed set \( X \) is nonempty almost surely, then the class \( S(X) \) is non-void too. Moreover, \( X \) coincides with the closure of a certain countable collection of its selectors. This collection is called the Castaign representation of \( X \).

Let \( m \) be a probability metric on the space of distributions of random vectors in \( \mathbb{R}^d \). Then the metric \( m_H \) on the space of random sets distributions is introduced in the same way as the Hausdorff metric \( \rho_H \) is defined by the Euclidean metric \( \rho \) in \( \mathbb{R}^d \). Put
\[
m_H(X, Y) = \max \left\{ \sup_{\xi \in S(X)} \inf_{\eta \in S(Y)} m(\xi, \eta), \sup_{\eta \in S(Y)} \inf_{\xi \in S(X)} m(\xi, \eta) \right\}.
\]

It is easy to show that \( m_H \) is a probability metric on the space of random sets distributions. Moreover, \( m_H \) inherits the homogeneous property of \( m \). Namely, if \( m \) is homogeneous of degree \( \gamma \), i.e.
\[ m(c\xi, c\eta) = |c|^{-\gamma} m(\xi, \eta), \quad c \neq 0, \]
then \( m_H \) is homogeneous too. Indeed, the class \( S(cX) \) coincides with \( cS(X) \), whatever \( c \neq 0 \) may be.

**Example 1.1** Let \( m \) be the simple engineering metric, i.e. \( m(\xi, \eta) = \rho(\mathbb{E}_\xi, \mathbb{E}_\eta) \). Then
\[
m_H(X, Y) = \max \left\{ \sup_{\xi \in S(X)} \inf_{\eta \in S(Y)} \rho(\mathbb{E}_\xi, \mathbb{E}_\eta), \sup_{\eta \in S(Y)} \inf_{\xi \in S(X)} \rho(\mathbb{E}_\xi, \mathbb{E}_\eta) \right\}
= \max \left\{ \sup_{x \in \mathbb{E}_X} \inf_{y \in \mathbb{E}_Y} \rho(x, y), \sup_{y \in \mathbb{E}_Y} \inf_{x \in \mathbb{E}_X} \rho(x, y) \right\}
= \rho_H(\mathbb{E}X, \mathbb{E}Y),
\]
i.e. in this case \( m_H \) coincides with the Hausdorff distance between the corresponding expectations of random sets. As in Section 2.1, \( \mathbb{E}X \) designates the Aumann expectation of the random set \( X \).