Chapter 4

Cauchy-Heine Transform

If \( \psi(w) \) is a continuous function on the straight line segment from 0 to a point \( a \) (\( \neq 0 \)), then

\[
f(z) = \frac{1}{2\pi i} \int_0^a \psi(w)(w-z)^{-1} dw
\]

will be analytic for \( z \) in the complex plane with a "cut" from 0 to \( a \), but generally, \( f(z) \) will be singular at the origin (and at \( z = a \)).

We will see that \( f(z) \) will have an asymptotic power series expansion (of Gevrey order \( k > 0 \)) at the origin, provided \( \psi(w) \) is "asymptotically zero" (of order \( k \)) as \( w \to 0 \). Hence integrals of the above type provide an excellent tool for making up examples of functions with asymptotic expansions or of series which are \( k \)-summable in certain directions; this will become clearer very soon.

4.1 Definition and Basic Properties

If \( k > 0 \) and a sector \( S \) are given, we write

\( A^{(0)}_k(S) \)

for the set of \( \psi \in A_k(S) \) with \( J(\psi) = 0 \), i.e. the set of analytic functions \( \psi \) (in \( S \)), such that to every closed subsector \( S_1 \) of \( S \) there exist \( c_1, c_2 > 0 \) for which

\[
|\psi(z)| \leq c_1 \exp\{-c_2|z|^{-k}\}, \quad z \in S_1
\]

(compare Ex. 3, Section 2.2). Let \( \psi \in A^{(0)}_k(S) \), \( S = S(d, \alpha, \rho) \), and fix \( a \in S \). Then the function

\[
f(z) = CH_a(\psi)(z) = \frac{1}{2\pi i} \int_0^a \psi(w)(w-z)^{-1} dw
\]

will be called Cauchy-Heine Transform of \( \psi(w) \) (with integration along the straight line segment). Clearly, \( f(z) \) is analytic for \( z \) (on the Riemann surface of the Logarithm) with

\[
\arg a < \arg z < 2\pi + \arg a ,
\]
and vanishes as $z \to \infty$ (if $f(z)$ even is analytic at $\infty$, if we consider $z$ in the complex plane instead of the Riemann surface, but that is of no importance right now). By deforming the path of integration, we can analytically continue $f(z)$ into the sector $\tilde{S} = S(d, \tilde{a}, \tilde{\rho})$, with $d = d + \pi$, $\tilde{a} = a + 2\pi$, $\tilde{\rho} = |a|$

**Proposition 1.** Let $S, \psi, \tilde{S}, f$ be as above. Then

$$f(z) \sim_k \hat{f}(z) \quad \text{in} \quad \tilde{S},$$

with $\hat{f}(z) = \sum_{n=0}^{\infty} f_n z^n$ so that

$$f_n = \frac{1}{2\pi i} \int_0^a \psi(w) w^{-n-1} dw, \quad n \geq 0.$$

Moreover, if both $z$ and $ze^{2\pi i}$ are in $\tilde{S}$ (i.e. $z \in S$ and $|z| < |a|$), then

$$f(z) - f(ze^{2\pi i}) = \psi(z).$$

**Proof.** For $N \geq 0$,

$$(w - z)^{-1} = \sum_{n=0}^{N-1} z^n w^{-n-1} + z^N w^{-N} (w - z)^{-1},$$

hence (with $f_n$ as above)

$$r_f(z, N) = \frac{1}{2\pi i} \int_0^a \psi(w) w^{-N} (w - z)^{-1} dw, \quad z \in \tilde{S},$$

(if we integrate according to $z$). For each closed subsector $\overline{S}_1$ of $\tilde{S}$ of opening less than $2\pi$ (larger subsectors can be split into finitely many pieces), one can choose a path of integration from 0 to a, so that $c = c(\overline{S}_1) > 0$ exists for which

$$|w - z| \geq c|w|,$$

for every $w$ on the path and every $z \in \overline{S}_1$. Since $\psi \in A_k^{(0)}(S)$, we have for sufficiently large $C, K > 0$ (independent of $w$)

$$|w^{-N} \psi(w)| \leq CK^N \Gamma(1 + N/k),$$

for every $N \geq 0$ and every $w$ on the path of integration. This implies (with $L$ being the length of the path of integration)

$$|r_f(z, N)| \leq c^{-1} CL(2\pi)^{-1} K^{N+1} \Gamma(1 + (N + 1)/k),$$

for every $N \geq 0$ and $z \in \overline{S}_1$, from which follows $f(z) \sim_k \hat{f}(z)$ in $\tilde{S}$. To prove the remaining identity, we observe

$$f(z) - f(ze^{2\pi i}) = \frac{1}{2\pi i} \int_0^a \psi(w) (w - z)^{-1} dw$$

with a closed path of positive orientation around $z$, hence Cauchy’s Formula completes the proof. \qed