

INTRODUCTION. According to a suggestion of Prof. P.A. Meyer, I have collected in this paper a number of interesting inequalities concerning operators. I have tried to include useful results, choosing in the literature the simplest proofs.

The author thanks Prof. P.A. Meyer for his careful reading of preliminary versions of the paper, pointing out several mistakes and simplifying some proofs.

§1. Operator-monotone and operator-convex functions

We denote by $\mathcal{H}$ a complex Hilbert space with scalar product $<\cdot,\cdot>$. In this section we assume $\mathcal{H}$ is finite dimensional, leaving to the reader the extension to (bounded) operators on an infinite dimensional space. We assume the reader is familiar with elementary definitions as positivity, spectrum, trace, etc.

The definition of a continuous function which is monotone non-decreasing (abbreviated below to monotone) or convex on self-adjoint operators is clear, and recalled below. Such a function is of course monotone (convex) in the ordinary sense, but this is far from sufficient. The most important result is Löwner's theorem ([30], 1934) which gives an explicit form for the operator monotone (convex) functions.

We denote by $T$ some interval of $\mathbb{R}$ and by $\text{Sp}^{-1}(T)$ the set of all operators $A$ whose spectrum $\text{Sp}(A)$ is contained in $T$. These operators are self-adjoint, and the description of the set $\text{Sp}^{-1}(T)$ ($\text{Sp}(A) \subseteq [a,b]$) shows that it is convex.

**DEFINITION.** A real (Borel) function $f$ defined on $T$ is called operator-monotone if for (any finite-dimensional Hilbert space $\mathcal{H}$ and) any two operators $A \leq B \in \text{Sp}^{-1}(T)$ on $\mathcal{H}$, we have $f(A) \leq f(B)$. It is called operator-convex, if for any two operators $A, B \in \text{Sp}^{-1}(T)$, we have

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B), \quad (0 \leq \lambda \leq 1).$$

If $f$ is monotone or convex in $T$ it is so in a smaller interval. On the other hand it is monotone or convex in the ordinary sense, hence locally bounded. Therefore it can be regularized by convolution in the usual way, remaining monotone (convex) on a slightly smaller interval. It will be convenient at some places to deal with $C^1$ or $C^2$ functions, but the results extend to full generality.

Here is the main theorem in this section. We break it into three statements for convenience.
THEOREM 1.1 (Löwner [30]). For every operator-monotone function \( f \) on \((-1,1)\), there exists a (unique) probability measure \( \mu \) on \([-1,1]\) such that

\[
f(t) = f(0) + f'(0) \int_{-1}^{1} \frac{t}{1-xt} \, d\mu(x).
\]

THEOREM 1.2. If \( f \) is operator-convex on \( T = [-1,1] \) and \( f(0) = 0 \), then \( g(t) = f(t)/t \) is operator-monotone on \( T \) (and conversely).

It follows that:

THEOREM 1.3. For each operator-convex function \( f \) on \( T = (-1,1) \), there exists a (unique) probability measure \( \mu \) on \([-1,1]\) such that

\[
f(t) = f(0) + f'(0) t + \int_{-1}^{1} \frac{tx}{1-xt} \, d\mu(x).
\]

It follows in particular that operator-monotone or convex functions are real analytic, and can be extended analytically outside \( T \). But we will not discuss this important topic (see Donoghue [15]).

There are several proofs of this celebrated theorem, see [1], [7], [13], [15], [22], [26], [30], [37] etc. Three remarkable proofs due to Löwner [30], Bendat and Sherman [7] and Korányi [26] are included in the book [15]. The proof we give here is adapted from the last remark in [22], where it is given as a simplification of Korányi's proof.

Example. We begin by an example of operator-monotone function which will show the sufficiency of (1.2). First take two operators \( 0 \preceq a \preceq b \), and \( A \succ 0 \). Then we have \( 0 \leq \lambda + a \leq \lambda + b \), implying \( I \preceq (\lambda + a)^{-1/2}(\lambda + b)(\lambda + a)^{-1/2} \). Taking inverses we get \( I \succeq (\lambda + a)^{1/2}(\lambda + b)^{-1}(\lambda + a)^{1/2} \) and finally the function \( f(t) = 1/(\lambda + t) \) is operator-decreasing. Then \( 1 - \lambda f(t) = t/(\lambda + t) \) is operator-monotone on \( T = [0,\infty[ \) and the same follows for any homographic function which is increasing on \( T \), and maps \( T \) into itself.

It follows that the mapping \((t-1)/(t+1)\) is a monotone increasing 1-1 mapping from \( \text{Sp}^{-1}(0,\infty) \) onto \( \text{Sp}^{-1}[-1,1]\). Carrying the result to the new interval we find that homographic increasing maps of \([-1,1]\) into itself are operator-monotone. This is the case for \( t/(1-xt) \) with \( x \in ]-1,1[ \), and it follows that (1.2) is indeed operator-monotone.

First characterization of monotone functions. Recall that the Hadamard product of two matrices (not operators!) \( A = (a_{ij}) \), \( B = (b_{ij}) \) is the matrix \( A \circ B = (a_{ij}b_{ij}) \). Shur's well known theorem asserts that the Hadamard product of a given matrix \( A \) with an arbitrary positive matrix \( B \) is positive if and only if \( A \) is positive.

We use in the whole paper the notation

\[
D_k f(A,H) = \frac{d^k}{dt^k} f(A+tH),
\]

whenever the right hand side exists. When \( t \) is omitted it is meant that \( t = 0 \).