Introduction

Let $\mathcal{X}$ be the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ such that $a > b \geq 0$ and $d > c \geq 0$. Let us consider the following series in $\mathbb{C}[M_2(\mathbb{Z})][[q]]$:

$$\sum_{M \in \mathcal{X}} M q^{\det M}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} q$$

$$+ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} q^2$$

$$+ \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} q^3$$

$$+ \ldots$$

The aim of this paper is to establish that this series produces (in a sense that will be made precise in a moment) Fourier expansions at infinity of modular forms of integral weight $\geq 2$ for congruence subgroups of $SL_2(\mathbb{Z})$. This justifies the terminology “universal Fourier expansions of modular forms”. Here and in what follows “Fourier expansion” will always mean “Fourier expansion at infinity”.

Let $k$ be an integer $\geq 2$. Let $N$ be an integer $> 0$. Let $C_{k-2}[X,Y]$ be the complex vector space of homogeneous polynomials in two variables and degree $k - 2$. Let $\Gamma_0(N)$ (resp. $\Gamma_1(N)$) be the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $N|c$ (resp. $N|(a-1)$). Let $\chi$ be a Dirichlet character modulo $N$. We denote by $S_k(N)$ (resp. $S_k(N,\chi)$) the complex vector space of cusp forms of weight $k$ for $\Gamma_1(N)$ (resp. for $\Gamma_0(N)$ with multiplicative character $\chi$, see section 2.5).

Let $C_{k-2}[X,Y][[\mathbb{Z}/N\mathbb{Z})^2]$ be the vector space of linear combinations of elements of $(\mathbb{Z}/N\mathbb{Z})^2$ with coefficients in $C_{k-2}[X,Y]$. If $\phi$ is a linear map $C_{k-2}[X,Y][[\mathbb{Z}/N\mathbb{Z})^2] \to \mathbb{C}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$, we denote by $\phi|_g$ the linear map $C_{k-2}[X,Y][[\mathbb{Z}/N\mathbb{Z})^2] \to \mathbb{C}$ defined by the formula

$$\phi|_g(P(X,Y)[u,v]) = \phi(P(aX + bY, cX + dY)[au + cv, bu + dv]),$$
$P \in C_{k-2}[X,Y]$, $(u,v) \in (\mathbb{Z}/N\mathbb{Z})^2$. We denote by $E_N$ the set of elements $(u,v)$ of $(\mathbb{Z}/N\mathbb{Z})^2$ satisfying the relation $zu + zv = Z/N\mathbb{Z}$.

Let us denote by $P_N = \cup_{d|N}(\mathbb{Z}/d\mathbb{Z})^*$. By convention when $d = 1$, $(\mathbb{Z}/d\mathbb{Z})^*$ has one element. Let $C[P_N]^k$ be the quotient vector space of $C[P_N]$ modulo the vector space generated by the elements of the form $[a] - (-1)^k[-a] = 0$, $a \in (\mathbb{Z}/d\mathbb{Z})^*$, $d|N$. If $a \in Z/N\mathbb{Z}$, $d|N$ and $a$ invertible modulo $d$, we denote by $[a]_d$ the image of $[a \pmod{d}]$ in $C[P_N]^k$.

Let $b : C_{k-2}[X,Y][E_N] \to C[P_N]^k$ be the $C$-bilinear map which associates to $P(X,Y)[u,v]$ the element $P(1,0)[u^{-1},v,N) - P(0,1)[-u^{-1},u,N)$, where by abuse of notations $v^{-1}$ is the inverse modulo $(u,N)$ of $v$ and $(u,N)$ is the greatest common divisor of $u$ and $N$, i.e. the order of the subgroup of $\mathbb{Z}/N\mathbb{Z}$ generated by $u$.

**Theorem 1** Let $\phi$ be a linear map $C_{k-2}[X,Y][(\mathbb{Z}/N\mathbb{Z})^2] \to C$ verifying the following equalities

$$\phi + \phi_\sigma = \phi + \phi_\tau + \phi_\tau z = \phi - \phi_J = 0,$$

where $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\phi(P[u,v]) = 0$ if $(u,v) \notin E_N$, $P \in C_{k-2}[X,Y]$. Let $z \in C_{k-2}[X,Y][E_N]$ such that $b(z) = 0$. Then

$$\sum_{M \in \mathcal{X}} \phi_M(z)q^{\delta + 1}M$$

is the Fourier expansion of an element $f$ of $S_k(N)$. Furthermore all modular forms of such type can be produced by this method.

Let $\chi$ be a Dirichlet character $\mathbb{Z}/N\mathbb{Z} \to C$. Let us suppose that $\phi$ satisfies the additional condition

$$\phi(P[\lambda u, \lambda v]) = \chi(\lambda)\phi(P[u,v])$$

$((\lambda, u, v) \in (\mathbb{Z}/N\mathbb{Z})^3$, $P \in C_{k-2}[X,Y])$. Then $f$ belongs to $S_k(N,\chi)$.

If $z$ does not satisfy the relation $b(z) = 0$, the series obtained should be, except for the constant term, the Fourier expansion of a holomorphic modular form of weight $k$ for $\Gamma_1(N)$ (see the remark in the section 3.2). This theorem can be refined to obtain only newforms (see sections 2.6 and 3.2).

We outline now the plan of the paper as well as the plan of the proof of the theorem 1. We recall in the first part the theory introduced by Manin and developed by Shokurov of modular symbols of arbitrary weight for subgroups of finite index of $S\ell_2(\mathbb{Z})$ ([4], [12], [11], [14], [13]). This theory can be related to the Eichler-Shimura theory connected with the cohomology of subgroups of finite index of $S\ell_2(\mathbb{Z})$. The Eichler-Shimura theory imbeds modular forms in a space of modular symbols; The Manin-Shokurov theory constructs pairings between modular symbols and modular forms. We construct a complex vector space (of modular symbols) $M_k(N)$, which is a quotient of $C_{k-2}[X,Y][E_N]$ (see section 1.3). We denote by $[P,z]$ the image of $P[z] \in C_{k-2}[X,Y][E_N]$ in $M_k(N)$. There is a bilinear pairing of complex vector spaces between $M_k(N)$ and $S_k(N) \oplus \overline{S_k(N)}$ (where $\overline{S_k(N)}$ is the space of antiholomorphic cusp forms), see section 1.5. This pairing is given as follows:

$$(f_1 + f_2, [P,z]) \mapsto \int_0^\infty f_{g1}(z)P(z,1)dz + \int_0^\infty f_{g2}(z)P(\bar{z},1)d\bar{z},$$