A PLATE ANALOGY FOR PLANE IMCOMPRESSIBLE VISCOUS FLOW

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2.1 Introduction. Finite element methods have been applied by various authors to the Navier-Stokes equations for incompressible viscous flow, particularly the plane case. Formulations in terms of velocity and pressure, vorticity and stream function, and stream function only, have been tried with varying success. [See Ref. 1 which also gives an up-to-date bibliography]. Only the simplest of transient flows have been considered, most problems studied being steady state solutions at low Reynold's Number. Whether the resultant steady state solutions are possible would appear to require a study of the variation of a perturbation of the steady state with time. It is clear that any numerical procedure which leads to steady state solutions under conditions where turbulence is known to be physically inevitable must be suspect. On the other hand if it proves impossible to establish a steady state solution whatever the numerical strategy employed one can be certain that no physical steady state exists.

One class of problem for which the stream function method is very suitable is Stokes flow - that is slow viscous flow with neglect of inertia or \( R_e = 0 \). The plate analogy for this case allows the use of the very sophisticated software available for plate bending.

The analogy derives from the Navier-Stokes equations written in the form

\[
\Delta^4 \psi = \frac{\mu}{\rho} \left[ \frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial \psi}{\partial t} \right] \Delta^2 \psi
\]  

(2.1)

Thus when \( \rho = 0 \) we have the equilibrium equation for an unloaded plate,

\[
\Delta^4 \psi = 0
\]  

(2.2)

2.2 Stokes Flow Analogy. Corresponding to the strain energy \( U \) of the plate we have the expression for half the energy dissipation rate due to fluid viscosity which is, for unit thickness,

\[
U = \frac{1}{2} \mu \iint \left[ (u_x^2 + v_y^2) + 2u_y^2 + 2v_x^2 \right] \, dx \, dy
\]

\[
= \frac{1}{2} \mu \iint \left[ \left( \psi_{xx}^2 + \psi_{yy}^2 \right) - 4(\psi_{xx} \psi_{yy} - \psi_{xy}^2) \right] \, dx \, dy \]  

(2.3)
Comparing with the plate energy equation we have the correspondence,

\[ \mu = D = \text{flexural rigidity} \]
\[ 4 = 2(1 - \text{Poisson's Ratio}) \]

Poisson's Ratio = -1
Young's modulus = 0
Shear modulus = \( 12 \mu t^{-3} \)

where \( t = \) plate thickness.

With this information we may construct the analogies in Fig. 2.1.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Plate</th>
<th>Fluid</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi )</td>
<td>Displacement</td>
<td>Stream Function</td>
</tr>
<tr>
<td>( \frac{\partial \psi}{\partial n} )</td>
<td>Edge rotation</td>
<td>-Tangential velocity ((-V_s))</td>
</tr>
<tr>
<td>( \mu \left( \frac{\partial^2 \psi}{\partial n^2} - \frac{\partial^2 \psi}{\partial s^2} \right) )</td>
<td>Edge moment ( M_n )</td>
<td>-Edge shear stress ((-\sigma_{ns}))</td>
</tr>
<tr>
<td>( -\mu \left( \frac{\partial}{\partial n} \Delta^2 \psi + 2 \frac{d}{ds} \frac{\partial^2 \psi}{\partial n \partial s} \right) )</td>
<td>Edge shear ( Q_n )</td>
<td>( -\frac{d\sigma_{nn}}{ds} )</td>
</tr>
</tbody>
</table>

Fig. 2.1: Plate Analogy for Stokes Flow

The last correspondence is not obvious but, \( \int Q_n \partial \psi ds = \) virtual work of edge forces, and