1. Introduction. A series of counterexamples (Cohen [1], De Giorgi [2], Coorjian [3], Pliś [6-11]) has shown that uniqueness theorems for differential equations with non-analytic coefficients require much more restrictive conditions than those in Holmgren's uniqueness theorem. However, there is a considerable gap between these counterexamples and the uniqueness theorems available. In this paper we shall try to narrow the gap or at least make it well defined by making a systematic analysis of the scope of the constructions used in the counterexample.

Let $P(D)$ and $Q(D)$ be two partial differential operators with constant coefficients in $\mathbb{R}^n$, $D = -i \nabla/\partial x$ as usual, and let $H_N$ be a half space

$$H_N = \{ x \in \mathbb{R}^n ; \langle x, N \rangle \geq 0 \}.$$ 

We shall study perturbations of $P$ by the operator $Q$. The problem is to decide when there is a function $a$ such that the equation

$$P(D)u + a Q(D)u = 0$$

has a solution $u \in C^\infty(\mathbb{R}^n)$ with

$$\text{supp } u = H_N.$$ 

We wish $a$ to vanish when $\langle x, N \rangle = 0$ so that the operator $P(D)$ is not perturbed there. The answer may of course depend on the conditions placed on $a$. We shall examine the cases where $a$ is required to be analytic, $C^\infty$ or $C^j$ for some finite $j$. The main results are Theorems 2.2, 3.1, 3.7 and 4.1.

Most uniqueness theorems known for equations of the form (1.1) require that $u$ vanishes outside a set with a strictly convex boundary. A direct comparison with the counterexamples proved here is therefore not possible.
For this and other reasons it would be interesting to modify the constructions with $H_N$ replaced by a strictly convex set, compact sets being particularly important. However, we shall not consider this problem at all here.

2. **Analytic perturbations.** First we recall the situation for the unperturbed operator $P$:

**Theorem 2.1.** The equation $P(D)u = 0$ has a solution $u \in C^\infty(\mathbb{R}^n)$ with
\[ \text{supp } u = H_N \] if and only if $P_m(N) \neq 0$, where $P_m$ is the principal part of $P$.

The necessity follows from Helmgren's uniqueness theorem (see Hörmander [4, Theorem 5.3.1]), and the sufficiency is proved by integrating suitable exponential solutions ([4, Theorem 5.2.2]). Holmgren's uniqueness theorem also gives the implication $2) \Rightarrow 1)$ in the following

**Theorem 2.2.** The following conditions are equivalent if $\mathcal{D}H_N$ is non-characteristic with respect to $P$:

1) The order of $P$ is smaller than the order of $Q$.

2) The equation (1.1) has a solution $u \in C^\infty(\mathbb{R}^n)$ satisfying (1.2) for some analytic $a$ in $\mathbb{R}^n$ vanishing when $\langle x, N \rangle = 0$.

3) For any given integer $k$ the equation (1.1) has a solution $u \in C^\infty(\mathbb{R}^n)$ satisfying (1.2) for some analytic $a$ in $\mathbb{R}^n$ vanishing of order $k$ when $\langle x, N \rangle = 0$.

**Proof.** Since $3) \Rightarrow 2) \Rightarrow 1$ we just have to prove that $1) \Rightarrow 3)$. Let $m$ be the order of $Q$. If $Q_m(N) \neq 0$, that is, $\mathcal{D}H_N$ is non-characteristic with respect to $Q$, the proof is somewhat simpler so we consider this case first. Choosing coordinates with $\langle x, N \rangle = x_1$ and taking $a$ and $u$ as functions of $x_1$ only, we find that it is then sufficient to prove the theorem in the one-dimensional case. Thus we assume that $n = 1$ and set with a positive integer $k$ and